An introduction to cosmology

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Abstract

These lecture notes give an introduction to cosmology for high-energy physicists. The content is otherwise orthodox. I introduce the cosmological principle, the basic equations, and the data that underly our understanding of the universe. Then I focus on the early universe, discuss nucleosynthesis, the WIMP paradigm of dark matter, and the principles of baryogenesis. The last two sections introduce elementary aspects of large-scale structure formation, and their relation to inflation. I apologise for not giving much reference to the original literature, but I point the reader to other reviews or lecture notes for more details or insights.

1 The Cosmological Principle

When we watch the sky at night, we see stars, the Milky Way, etc. But if we blur the picture a little, we would agree that there seems to be no preferred direction. This isotropy together with the alleged Copernician principle —which states that the Earth holds no special place— has led to assume that the Universe is homogeneous on large scales. This hypothesis, called the *Cosmological Principle*, has been central to the development of modern cosmology, starting with the static universe of Einstein which, he assumed, should be uniform both in space and in time (see for instance Ref.[1] and [2] for history).

Isotropy and homogeneity are clearly different concepts. There are systems which are isotropic but not homogeneous and the other way around. However, isotropy around any two points implies homogeneity, as Fig. 1 suggests.

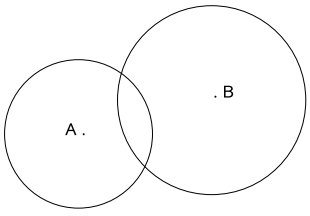

Fig. 1: Isotropy around A and B means that physical conditions (say density or temperature) are the same on the two circles and thus on any circle and so implies homogeneity.

The Cosmological Principle is well supported by observations. In particular:

- The isotropy of cosmological signals (most remarkably the cosmic microwave background radiation (CMBR)).
- The large-scale distribution of matter (large-scale structures).
- The recession of distant galaxies (Hubble's law).

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The CMBR, discovered in 1965, is a spectrum of electromagnetic radiation that peaks in the microwave range ($\lambda \approx 2$ mm) and that fills the universe. In the early 1990s, the FIRAS instrument on board ot the COBE satellite established that this radiation has an almost perfect black body spectrum at temperature $T = 2.725$ K (see Fig. 2). This discovery brought to an end work on the *Steady-State model*, an alternative to the Big Bang. The CMBR signal is very isotropic. There is a dipole at the level $\Omega = \Delta T/T \sim 10^{-3}$

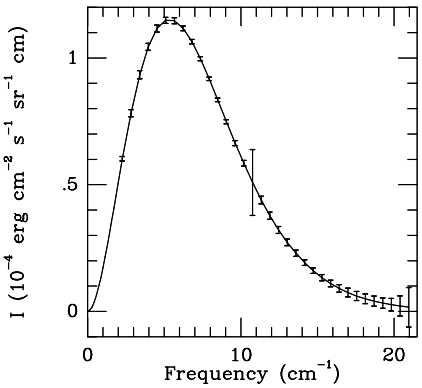


Fig. 2: The spectrum of the CMBR (FIRAS). Errors have been enlarged by a factor of 400.

that is interpreted as a Doppler effect caused by motion with respect to the frame of reference in which the CMBR signal is isotropic. There are also higher multipoles, but at a much smaller level, $\Omega \approx 10^{-5}$. Their significance will be discussed in Section 7.

Further evidence of isotropy is provided by studies of the distribution of galaxies (Fig. 3) or other extragalactic objects, like gamma-ray bursts (Fig. 4). Surveys in redshift, like the Las Campanas

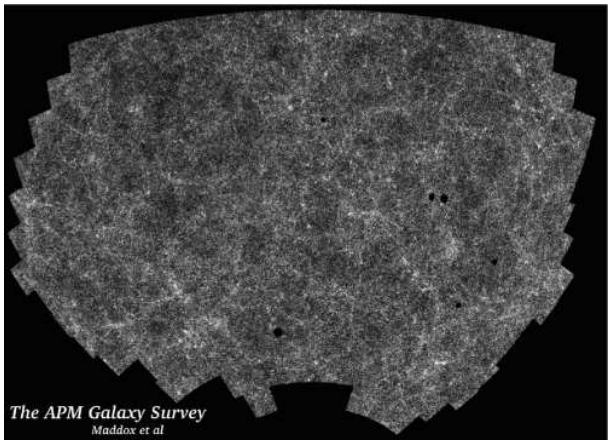


Fig. 3: The APM survey shows the distribution of density of galaxies in the sky (about 10^6 galaxies, the brighter the colour, the higher the density).

Redshift Survey or the 2dF survey (Fig. 5) show that the average distribution of galaxies is uniform. There are structures in these maps but the density contrast (defined $\Delta = \delta\rho/\rho$ where ρ is the energy density of matter) is $\Delta \ll 1$ on scales beyond ~ 100 Mpc [1]. This is shown in Fig. 6 using a compilation of (somewhat old) data [3].

The law of recession of galaxies was formulated by Hubble in 1929, using cepheid stars to mea-

[1]The parsec (1 pc $\approx$ 3.3 light-years) is a unit of distance much used in astronomy and still common in cosmology. Seen from an object at 1 pc, the Sun-Earth distance (1 AU = $150 \cdot 10^6$ km) would sustain an angle of 1 sec.

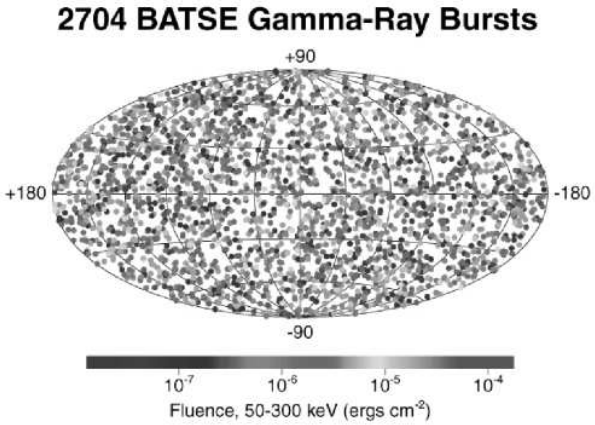


Fig. 4: The gamma-ray burst angular distribution measured by BATSE (http://www.batse.msfc.nasa.gov/batse/grb/)

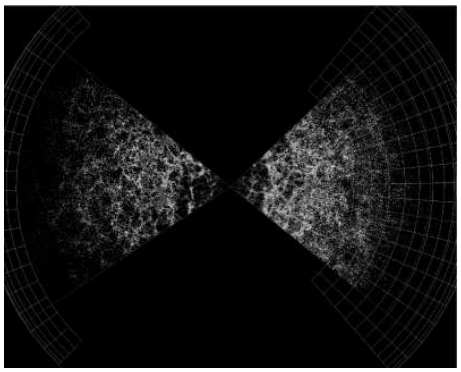


Fig. 5: Map of the 2dF redshift survey (http://www2.aao.gov.au/2dFGRS/)

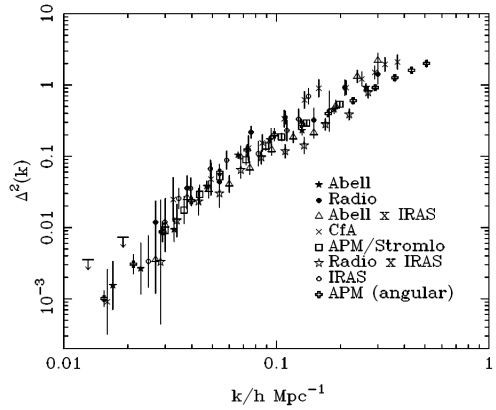


Fig. 6: Power spectrum of mass fluctuations $\propto \langle\Delta^2\rangle$, where h is as in (3).

sure cosmic distances. Measurements of spectra of galaxies showed a systematic increase of measured wavelengths λ_o with respect to those observed in the laboratory λ_e. Defining the redshift parameter z

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

gives

$$zc = H_0 d \tag{1}$$

where d is the distance to the observed galaxy, c is the speed of light, and $H_0 \approx 500$ km·s^{-1}· Mpc^{-1} is the Hubble parameter (the value quoted is that measured by Hubble). If the redshift is interpreted as a Doppler effect due to motion of the galaxy, $v/c \approx z$ for non-relativistic motion, we get the Hubble law,

$$v = H_0 d, \tag{2}$$

which states that galaxies recess with a velocity proportionnal to their distance. This motion is an average, as the velocity of galaxies at fixed distance are distributed around the mean given by the Hubble flow. The difference is called the peculiar velocity. For large distances, the ratio of the peculiar velocity to the Hubble velocity is small but for neighbouring galaxies the dispersion is important. Figure 7 shows

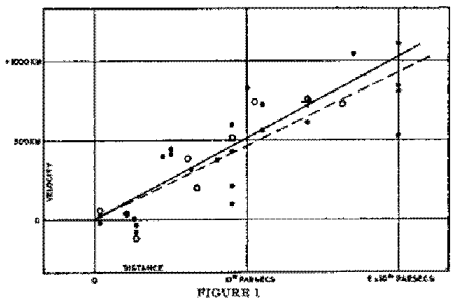


Fig. 7: The first Hubble diagram: $H_0 \approx 500\,\mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1}$.

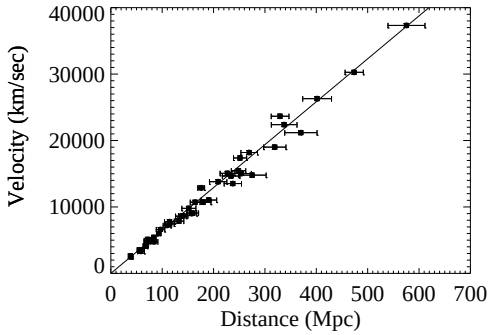


Fig. 8: A modern Hubble diagram: $H_0 = 64\,\mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1}$.

the historic data. Figure 8 is a contemporary Hubble diagram. The giant leap in distances between the two figures has been achieved by using a class of supernovae (called type Ia or SnIa for short) as standard candle to measure cosmic distances. It is the use of SnIa that has led to the conclusion that the expansion of the universe is accelerating (see Section 3). To find good candles has always been a problem in cosmology and SnIa are no exception (see, for instance, Ref.[4] or references in Ref.[5]).

In the sequel we will use

$$H_0 = h100\,\mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1} \text{ with } h \approx 0.7 \tag{3}$$

For estimates take $h \approx 2/3$ and $h^2 \approx 1/2$.

What is the relation between the Hubble law and the cosmological principle? There are two quite different interpretations of the Hubble law. The first one is that we are at the centre a sort of explosion and that the galaxies move away from us. The alternative interpretation is that there are no privileged observers. The system is uniform (there are an infinite number of centres) and galaxies are moving away from each others. The classical illustration is a balloon being inflated (we live on the surface of the balloon). Equivalently take a system of galaxies and assume that their their peculiar velocity is negligible (ideal galaxies). The position of these ideal galaxies defines a system of coordinates called comoving coordinates, that is a system of coordinates in which they are at rest. Motion is taken into account by introducing a scale factor $a(t)$ which depends only on time (which we will call the age of the universe). That the Hubble law holds is shown in Fig. 9. It is easy to verify that the $v \propto d$ is the only possible motion consistent with the cosmological principle (*e.g.*, $v \propto d^2$ would not work).

An immediate consequence of the Hubble law is that if we reverse the flow and go back in time there would be a time at which the galaxies were infinitely close to each other. The time scale for this is given by the inverse of the Hubble parameter H_0

$$1/H_0 = h^{-1}\,9.78\cdot 10^9\,\text{years}.$$

For $h \approx 2/3$, we get $1/H_0 \approx 15\cdot 10^9$ years, which is older than the age of the globular cluster ($\sim 12\cdot 10^9$ years) the oldest system of stars. In the days of Hubble, $1/H_0 \approx 2\cdot 10^9$ years, which was less than the age of the Earth (for a history of measurements of the Hubble parameter see, for example, [6]).

If v was constant (that is to say for a galaxy at comoving distance x), H^{-1} would indeed be the age of the universe, since

$$v = \text{const} = \dot{a}(t)x \to a(t) \propto t \to H(t) = \dot{a}(t)/a(t) = 1/t\,,$$

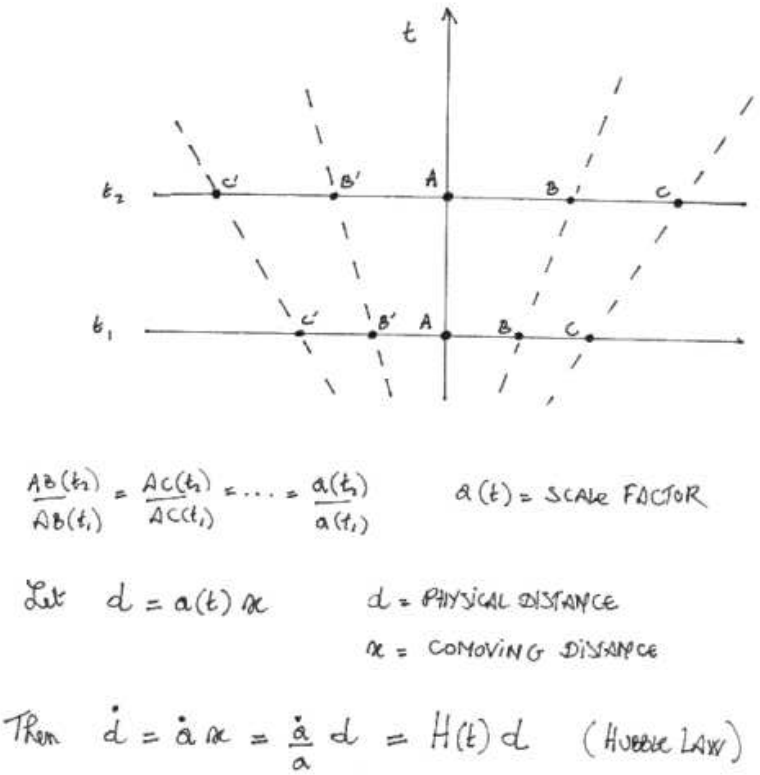


Fig. 9: A one-dimensional derivation of the Hubble law from the introduction of a uniform scale factor $a(t)$. Things are depicted from the point of A but the same result holds from the point of $B^{(\prime)}, C^{(\prime)}$, etc.

choosing the origin of time so that $a(0) = 0$. Constant velocity is free motion. Gravity is an attractive force and thus we should expect that attraction between galaxies will slow their collective motion. Consequently, the age of the universe should be less than the naive estimate $t_0 = 1/H_0$. To verify this, we need the equations that describes our system of galaxies, as well as everything else that might fill the universe.

2 Basic equations

We now write down the basic equations describing a perfectly homogeneous and isotropic expanding universe. Small perturbations will be discussed in Sections 7 and 8.

The fist thing we need is a convenient system of coordinates to describe the spacetime of our (idealized) universe. Spatial slices or sections in spacetime are taken to be isotropic and homogeneous. By definition physical conditions (say some energy density ρ) are constant on each slice. As in the previous section, we take the positions of ideal galaxies (*i.e.*, with no peculiar motion) to define comoving coordinates. Here we use spherical comoving coordinates (χ, θ, φ) and our galaxy is put at the origin of coordinates. As for time, we make use of the fact that the lapse of proper time measured by any ideal (last time) galaxy between a spatial slice with physical conditions A (say ρ_A) and B (ρ_B) is a universal quantity. Hence we take our proper time as a universal coordinate of time t and call it the 'age of the

universe'. The age today is written with the subscript *zero*, $t(today) = t_0$ (likewise H_0 is the Hubble parameter today). Finally the collective motion (Hubble flow) is implemented through the scale factor $a(t)$ that we normalize to $a(t_0) = a_0 = 1$ today. Hence the physical distance *today* between our galaxy and a galaxy B is given by χ_B, Fig. 10. For any time, the physical distance d_P is given by

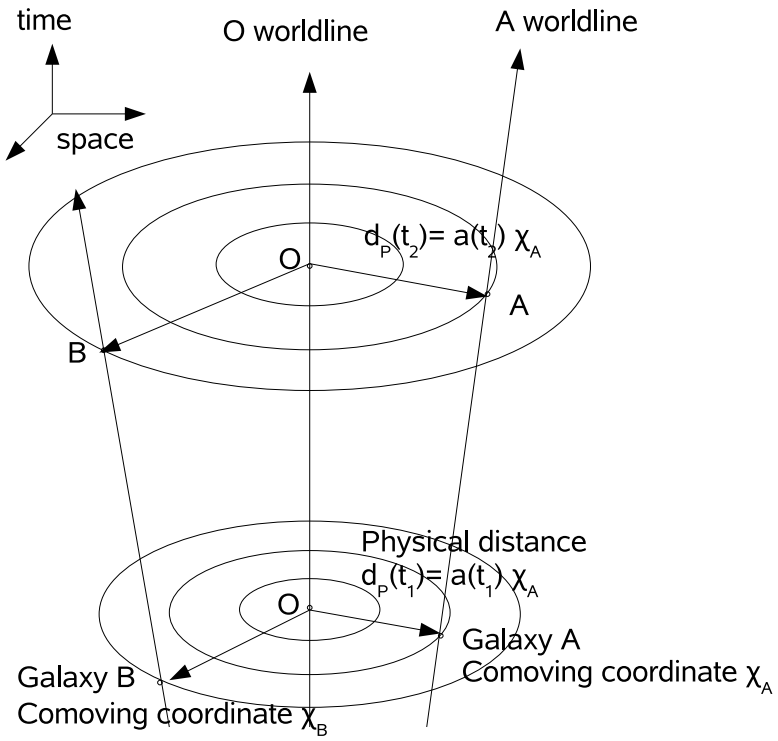


Fig. 10: System of comoving coordinates for ideal galaxies, *i.e.*, galaxies at rest in comoving coordinates. Since space is homogeneous, a universal time of coordinate ('age of the universe') is provided by the proper time of comoving observers. At each t, the physical distance between a galaxy (us) at the origin and another one is given by $a(t)\chi$.

$$d_P(t) = a(t)\chi\,.$$

Taking the time derivative gives the Hubble law

$$v = \dot{d}_P = \dot{a}(t)\chi = \frac{\dot{a}}{a}a(t)\chi \equiv H d_P$$

This relation between velocity and physical distance is *exact* (compare with (2) obtained from the observation of redshift and the non-relativistic expression of the Doppler effect) but holds only if we use the distance d_P (Section 3).

According to General Relativity (GR) —the framework that we should really use to describe the universe as a whole— energy/matter curves spacetime and, conversely, the shape of spacetime tells

energy/matter how to move. The basic building block of GR is the metric of spacetime. This metric takes a very simple form for an isotropic and uniform universe. Using our coordinates (t,χ,θ,φ), an infinitesimal spacetime interval reads

$$ds^2 = dt^2 - a(t)^2 dl^2\,,$$

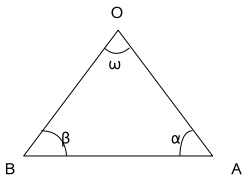
where dl is an infinitesimal comoving distance interval. The latter has to be consistent with the isotropy and homogeneity of space[2]. Correspondingly the geometry of a space slice does not have to be euclidean. Actually there exists three isotropic and homogeneous geometries and the corresponding spacetime metrics are called the Robertson-Walker metrics[3]. The spatial geometries correspond to the equivalent in three dimensions of the sphere, the plane, and the hyperbolic plane. These are surfaces of constant curvature, noted K (for a sphere of radius R, $K = 1/R^2$) and

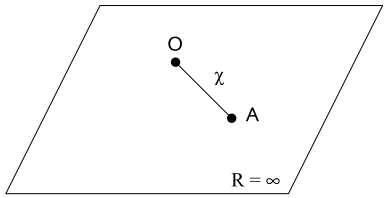
$$dl^2 = d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)$$

with (drawing the analogs of the geometries in two dimensions for the sake of illustration)

- Flat space: $K = 0$

$$S_K(\chi) = \chi$$

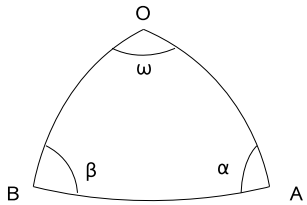


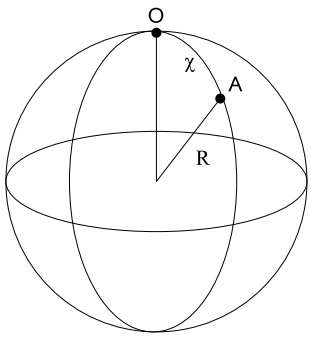


$$\alpha + \beta + \gamma = \pi;$$

- Spherical space: $K > 0$

$$S_K(\chi) = \frac{1}{\sqrt{K}} \sin\sqrt{K}\chi$$

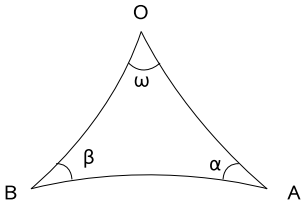


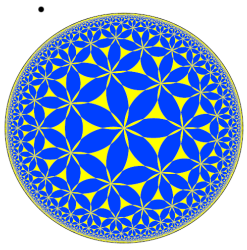


$$\alpha + \beta + \gamma > \pi;$$

- Hyperbolic space: $K < 0$

$$S_K(\chi) = \frac{1}{\sqrt{-K}} \sinh\sqrt{-K}\chi$$





$$\alpha + \beta + \gamma < \pi\,.$$

[2]Here we make a conceptual jump: we no longer think of galaxies in motion but rather we shall interpret the Hubble flow as being caused by the expansion of the universe itself.

[3]Standard caveat: we discuss only the geometry of space, not its topology. For instance, space could be flat, but curled, like a three-dimensional torus. This may be, but we assume here that the radii of such a torus are much larger than the largest distance yet accessed in our universe —to be called the horizon.

These geometries are all isotropic and homogeneous. This is clear for the plane and the sphere. In the latter case curvature is just $K = 1/R^2$ where R is the radius of the sphere (remember, we live *on* the sphere). The plane corresponds to the limit $R \to \infty$. The hyperbolic plane is less obvious but it really is the same as the sphere if we take an imaginary radius $R \to iR$ (see also the Escher-like drawing. This is the Lobachevsky represtation of the hyperbolic plane)

A nice feature of the coordinate system introduced here is that the distance today between us and a distant galaxy B is simply given by its comoving coordinate χ_B, regardless of the geometry. Using χ we may ask and answer interesting questions, like how long it would take for light to travel from such a galaxy to us. Since light travels on the lightcone we have

$$d\chi^2 = dt^2/a(t)^2$$

since the light travels towards us along line of constant (θ_B, φ_B), and

$$\chi_B = \int_{t_e}^{t_o} \frac{dt}{a(t)},$$

where t_o is the time of observation and t_e is the time of emission. Now consider that the light emitted has wavelength λ_e at emission and wavelength λ_0 at observation. The periods at emission $T_e = \lambda_e/c$ and observation $T_o = \lambda_o/c$ satisfy

$$\chi_B = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+T_e}^{t_o+T_o} \frac{dt}{a(t)}.$$

Since the period is much (much) smaller than the travel time (the nearest galaxy, Andromeda, is at about 1 Mpc away while T is $\sim 10^{-15}s$ for visible light) we get, after reorganizing the terms of the integrals,

$$\int_{t_e}^{t_e+T_e} \frac{dt}{a(t)} = \int_{t_o}^{t_o+T_o} \frac{dt}{a(t)} \longrightarrow \frac{T_e}{a(t_e)} = \frac{T_o}{a(t_o)} \quad \text{for} \quad \frac{T}{t} \ll 1.$$

With $a(t_o) = a(t_0) = 1$,

$$\frac{\lambda_o}{\lambda_e} = \frac{1}{a(t_e)} \equiv 1 + z,$$

where z is the redshift factor derived in Section 1. We see that light observed with, say, redshift $z = 1$ was actually emitted when the universe was a factor of $(1+z) = 2$ smaller.

So far our discussion is purely kinematical and we do not know yet the actual time dependence of $a(t)$. We have already discussed two ingredients of the universe: galaxies and light. In more general the matter/content of an isotropic and homogeneous universe is a sum of perfect fluids, characterized by their energy density $\rho(t)$ and pressure $p(t)$. Depending on the context, a fluid may be a gas of non-relativistic or relativistics particles. The former includes a set of galaxies treated as point particles, dust, dark matter,etc., while the former could be photons, neutrinos or any particle that is non-relavistic at a given moment.

Given the matter/energy content, the canonical path is to put it in the RHS,of the Einstein equations, through the stress-energy tensor, together with a Robertson-Walker metric. This gives two independent differential equations for $a(t)$, $\rho(t)$ and $p(t)$. Adding the equation of state that relates p and ρ gives a closed set of equations that may be solved for $a(t)$, $\rho(t)$, etc.

We take here a different path, that relies as much as possible on our intuition of newtonian dynamics. The good news is that the equations we obtain this way are exactly those that GR would give, at least for the case of a non-relativistic fluid.

We consider a spherical region of the universe centred on a given point (let us say us, but really it does not matter) of radius d at time t and a uniform energy density ρ. Let us consider a test galaxy on the

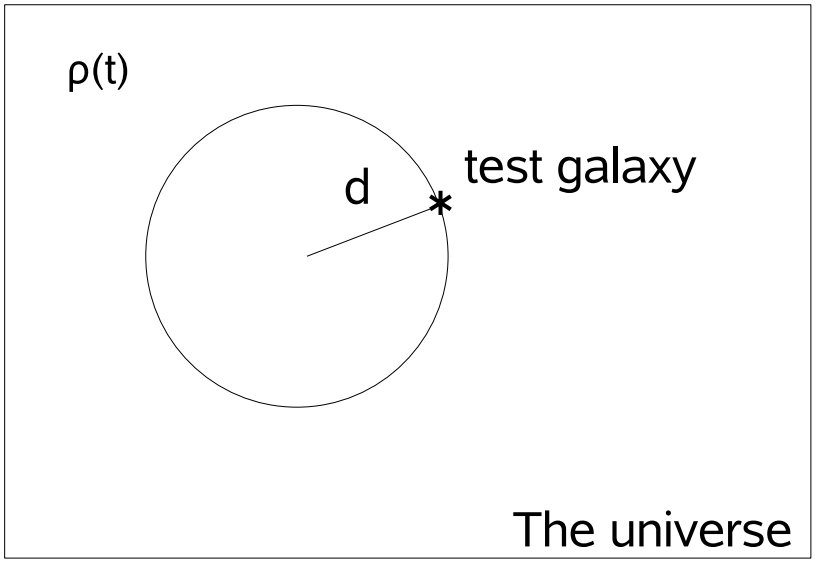


Fig. 11: In a uniform system, the motion of a test galaxy moving away from the given point is dictated by the mass within the sphere M centred on the given points.

sphere. Since the distribution of matter is isotropic around the centre of the sphere and since Newton's law of gravitation is $\propto 1/r^2$, only the mass M within the sphere exerts a net gravitational force on the test galaxy (see fig.11)[4]. If the motion of the galaxy is purely radial (we want to describe our universe in which galaxies have only radial motion), the equation of motion for the galaxy test is

$$m\ddot{d} = -\frac{GMm}{d^2} = -\frac{4\pi G}{3c^2} m\rho d\,,$$

where G is Newton's constant and c^2, the speed of light, is there because ρ is energy density, not mass density (later we set $c = 1$). Using $d = a\chi$ and simplifying we obtain the *Raychaudhuri equation*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\rho\,.$$

All reference to the origin and to the mass of the test galaxy have dropped and we have an equation for a. Moreover, it is the same as you would get from General Relativity, if the energy density were that of a non-relativistic fluid. On the way, we have learned that the effect of matter is to slow down the expansion of the universe, since $\ddot{a} < 0$ for $\rho > 0$.

If the fluid or gas is made of relativistic particles, we should take into account its pressure. For an ideal gas, pressure is a measure of mean kinetic energy and for a fluid of relativistic particles, ρ and p are of the same order. The equation of state of a gas of relativistic particles is

$$p = \frac{1}{3}\rho\,.$$

[4]This argument is fishy because the universe is *a priori* infinite and going to this limit is non-trivial. We should really use General Relativity here.

For non-relativistic particles or dust, pressure is negligible compared to the energy density (kinetic energy less than mass at rest) and the equation of state is simply

$$p \approx 0\,,$$

meaning $p \ll \rho$.

How does pressure enter the Raychaudhuri equation? This is clearly a relativistic correction, so we need General Relativity. The correct equation is

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,(\rho + 3p)}\,,$$

so the effective gravitational mass/energy is $\rho + 3p$ (the factor of 3 is there because there are three spatial dimensions). If $p > 0$, the pressure of a fluid is as attractive as its energy density[5]. More generally we see that expansion is *decelerated* as long as

$$\rho + 3p > 0\,.$$

This is the case for a gas of particles, both with relativistic or non-relativistic particles.

More general fluids are, however, considered by cosmologists, in general of the form $p = w\rho$. A particularly intriguing case is $w = -1$. For $\rho > 0$, this is a fluid with negative pressure. This seems odd at first but for a medium, negative pressure is like positive tension, a mundane property of materials like an elastic. The only difference is that, for the pressure/tension to be relevant, it has to be of same order as its mass/energy density (a relativistic elastic). The reasons such fluids are being considered is that a fluid with $p = -\rho$ gives accelerated expansion (see the footnote on pressure gradients). This is also called a cosmological constant (up to a factor) or, more recently, dark energy.

To solve for $a(t)$, we need another equation. This will be provided by conservation of energy of the fluid. The first law of thermodynamics applied to $E = \rho V$ with $V \propto a^3$ gives

$$dE \equiv \rho dV + V d\rho = -p dV + T dS\,.$$

A key feature of an homogeneous and isotropic universe is that expansion is adiabatic ($dS = 0$). This is a consequence of the Einstein equations but physically it comes because heat has nowhere to go in an homogeneous and isotropic system. Thus

$$\boxed{\dot{\rho} = -3H(\rho + p)\,.}$$

Applying this to the three kinds of fluid envisioned above we obtain:

- Dust, non-relativistic matter:

$$p = 0 \quad \longrightarrow \quad \rho \propto a^{-3}\,.$$

This is natural: expansion just dilutes the energy density.

- Radiation, relativistic matter:

$$p = \frac{\rho}{3} \quad \longrightarrow \quad \rho \propto a^{-4}\,.$$

For a relativistic fluid, dilution is faster than in a non-relativistic fluid.

[5] A possible misconception is that (positive) pressure should give repulsion. This is true when there is a gradient of pressure (like a gas in a balloon) but in an homogeneous universe, pressure is uniform and thus no gradient.

-Cosmological constant, dark energy:

$$p = -\rho \quad \longrightarrow \quad \rho = \text{const.}$$

For a cosmological constant (dark energy), the energy density stays constant. Hence the name.

The Raychaudhuri and energy conservation equation can be combined to get a first order equation for a, called the *Friedmann equation*.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}\,, \tag{4}$$

where $H = \dot{a}/a$. Going from a second order to a first order equation, we must introduce an integration constant. We have identified it with K, the curvature of space (we set $c = 1$). To establish this connection you need GR but we will make it more intuitive very soon.

This equation plays a central role in cosmology as it relates three important parameters: the Hubble constant, the total energy density of the universe, and its curvature, or geometry. Dividing the Friedmann equation by H^2, we get

$$1 = \Omega - \frac{K}{a^2H^2}\,, \tag{5}$$

where we have introduced the energy density parameter

$$\Omega = \frac{\rho}{\rho_c}\,,$$

where ρ_c is the *critical density* defined as

$$\rho_c = \frac{3H^2}{8\pi G}\,.$$

Today

$$\begin{aligned}
\rho_{c0} = \frac{3H_0^2}{8\pi G} &= 1.88\,h^2 \times 10^{-29}\,\mathrm{g\cdot cm^{-3}} \\
&= 1.1\,h^2\,10\,\mathrm{GeV\cdot m^{-3}} \\
&= 2.775\,h^2\,10^{11}\,\mathrm{M_\odot\cdot Mpc^{-3}} \\
&= (3\times 10^{-3}\,\mathrm{eV})^4\,h^2\,.
\end{aligned} \tag{6}$$

From Eq. (5), we see that $\Omega > 1$ corresponds to a universe with spherical geometry (sometimes called a 'closed universe'). If $\Omega < 0$, it is hyperbolical ('open'). If $\Omega = 1$, it is flat. No wonder that much observational effort is put into determining Ω.

Consider the third number in Eq. (6). It corresponds to having roughly one spiral galaxy like Andromeda per cubic Mpc, or on average about ten protons per cubic metre. This suggests that the energy density of our universe is not far from the critical density. However, despite the fact that most baryons are are invisible to the 'eye' in the universe, we know pretty well how many there are. It turns out that the energy density in baryons (ordinary matter) is substantially less than the critical energy density, as we shall see later in Section 4.

Let us consider yet another form of the Friedmann equation. If we multiply the Friedmann equation by $a^2/2$ we get

$$\frac{\dot{a}^2}{2} - \frac{4\pi G}{3}\rho(a)a^2 = -\frac{K}{2}\,.$$

If we interpret a as 'position', this equation is analogous to conservation of energy in one dimension $E =\equiv -K/2$ for a particle of unit mass moving in a potential $U \propto -\rho a^2$. For a non-relativistic fluid, $\rho \propto a^{-3}$ and thus $U \propto -1/a$. We may use our intuition of simple dynamical system to analyse the possible solutions, as depicted in Fig. 12. Originally a is close to zero and increasing in expanding solutions. The origin of time corresponds to $a = 0$, a singular solution since $\rho \to \infty$ at that moment. This singularity is called the Big Bang. If energy is negative, $K > 0$, expansion stops at some maximal scale factor a and then decreases. If $K < 0$, expansion lasts for ever. If K equals exactly zero (flat), expansion slows down and comes to rest asymptotically. This set-up is precisely analogous to the escape velocity problem in a gravitational system.

For a cosmological constant $U \propto -a^2$, a reversed harmonic oscillator, see Fig. 13.

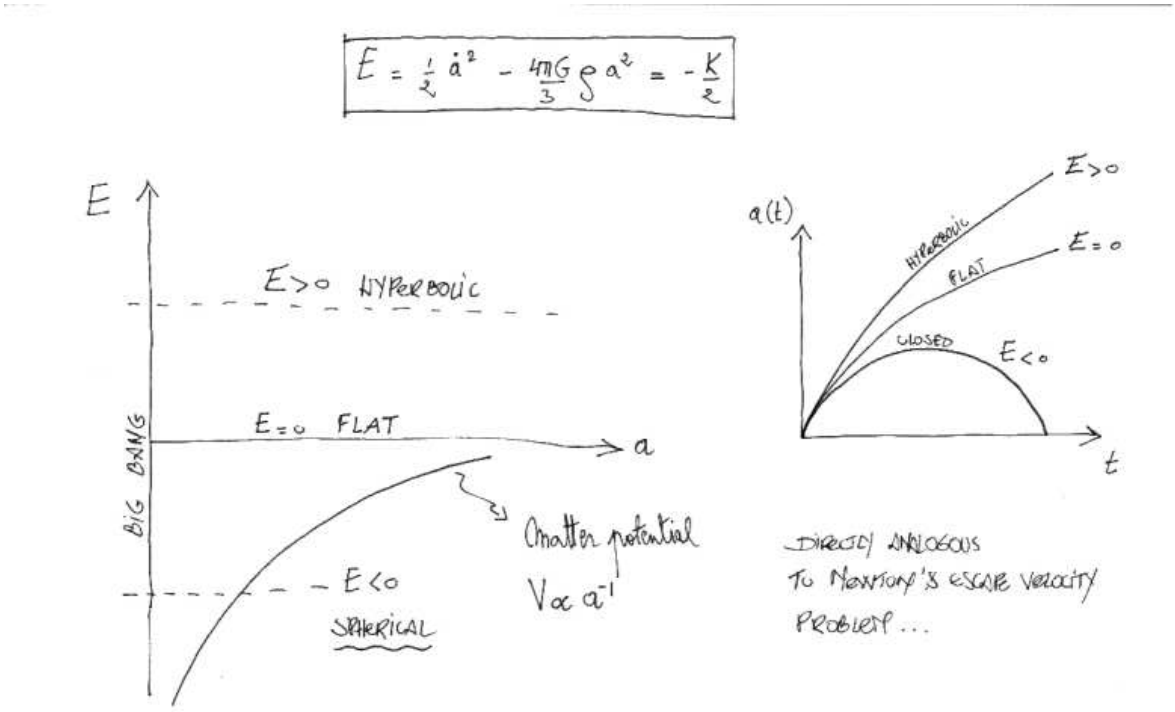


Fig. 12: for a matter-dominated universe, the dynamics of expansion is directly analogous to the escape velocity problem. If $K = 0$, $\dot{a}$ goes to zero at infinity. If $K > 0$ (more energy density), $\dot{a}$ vanishes at some point and then increases. This solution corresponds to a recollapsing universe, towards a big crunch.

Armed with our equations and their qualitative solutions, we may look for explicit solutions. You may check the solutions given in the table below for the simplest case of a flat univers, $K = 0$, when there is just one fluid. The solutions are normalized so that $a(t_0) = 1$.

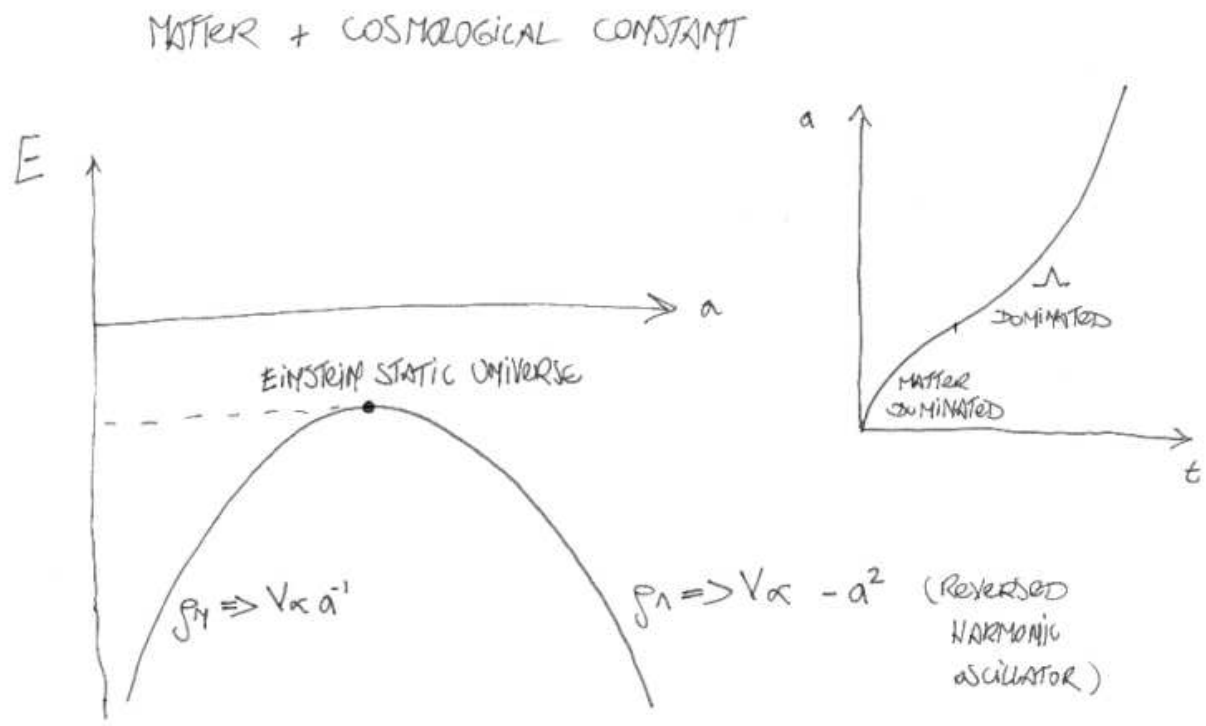


Fig. 13: If there is a mixture of matter and a cosmological constant, the potential shows a maximum (the static albeit unstable solution, at fixed a, was that found by Einstein and the reason why he introduced a cosmological constant). Interesting solutions (*i.e.* analogous to what we observe) have first decelerated expansion, when the energy density of matter is dominant, and then accelerated expansion, when the cosmological constant takes over.

-Matter

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \qquad H = \frac{2}{3t}$$

-Radiation

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \qquad H = \frac{1}{2t}$$

-Λ

$$a(t) = a(t_i)\exp(H(t-t_i)) \qquad H = \text{const}$$

The solution for a matter dominated universe is particularly interesting. We may compare to an empty universe $\rho \to 0$, thus with $K < 0$. In that limit,

$$a = t/t_0\,,$$

which might be called inertial expansion, and

$$t_0 = \frac{1}{H_0}\,.$$

In presence of matter, expansion is decelerated and the result is that the time that has lapsed since the Big Bang (the age of the universe) is shorter by a factor of $2/3$

$$t_0 = \frac{2}{3H_0}\,.$$

Using $H_0^{-1} \approx 3/2 \cdot 10^{10}$ years, gives

$$t_0 \approx 10^{10}\,\text{years}\,,$$

which is shorter than the age of the oldest globular clusters. Hence our universe can not be flat, matter dominated. This is an instance of an 'age crisis', a recurent issue in cosmology.

What about other solutions? Some are depicted in Fig. 14. They all have the same Hubble parameter today. They differ by their energy content. The most remarkable feature is that the introduction of a cosmological constant in an otherwise flat geometry gives an older universe. This is historically an important motivation for introducing a cosmological constant, to start with Einstein, who introduced the cosmological constant to get a static universe, $t_0 \to \infty$.

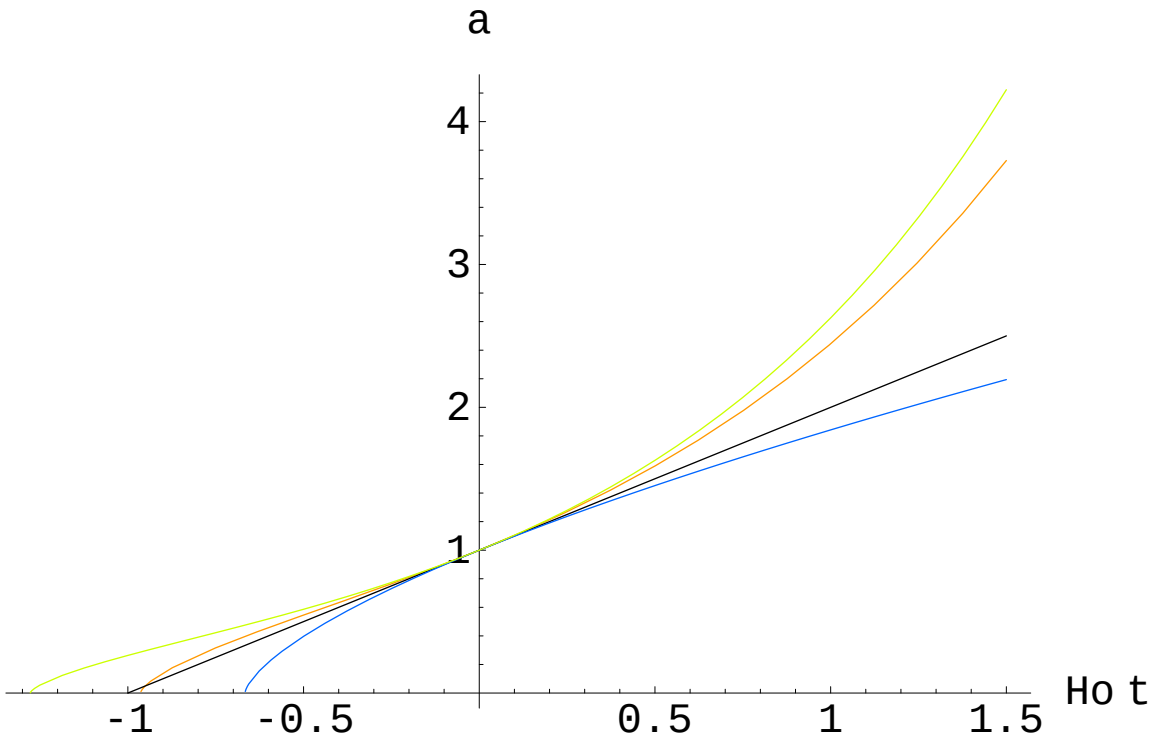


Fig. 14: From bottom to top: flat matter-dominated (blue), empty (black), 30% matter + 70% Λ (red), 10% matter + 90% Λ (yellow). The red curve is consistent with what we know of the composition of the universe.

A cosmological constant is not the only way to make the universe older. An open universe would be also older than a flat universe. An extreme instance with hyperbolic geometry is an empty universe $\rho \to 0$, for which $t_0 = 1/H_0$ and also displayed in Fig. 14. More generic solutions are displayed in the diagram of Fig. 15 which shows that an open universe is older than a flat universe with no cosmological constant (the star).

The solution currently favoured by data is the square, with $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$. In the next section, we review the evidence for the accelerated expansion of the universe.

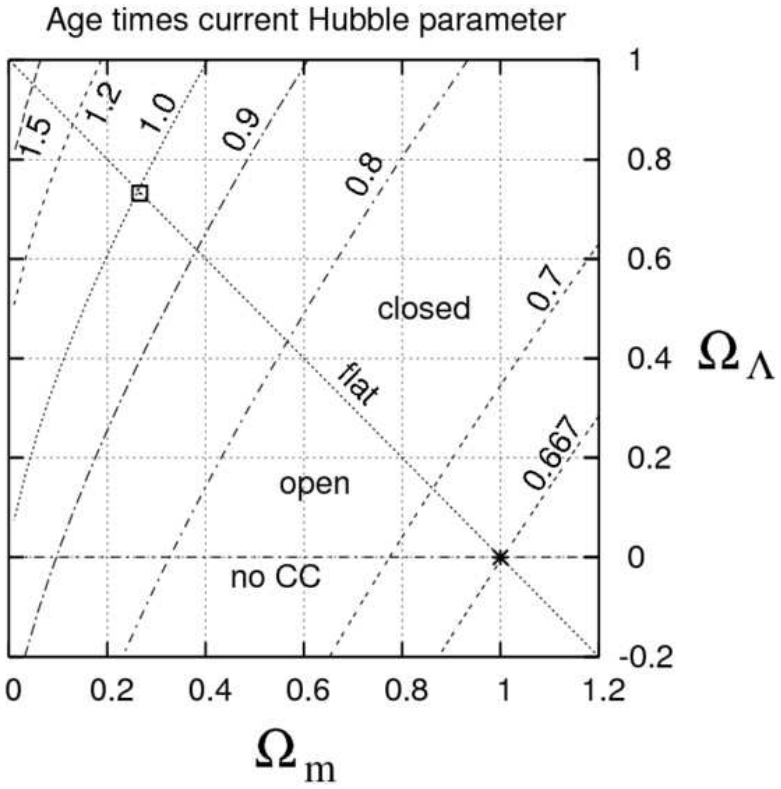


Fig. 15: Age of the universe as a function of its composition in matter and cosmological constant and of its geometry.

3 Mapping the cosmological expansion

Determining the composition of the universe is one of the most important problems in cosmology. One clear way is to map the cosmological expansion. Take the Friedmann equation (4) with different contributions to the energy density, say radiation, matter, and a cosmological constant

$$H^2 = \frac{8\pi G}{3}(\rho_r + \rho_m + \rho_\Lambda) - K/a^2 \,.$$

Using the dependence of these energy densities on the scale factor a, or equivalently on the redshift z, we may rewrite this equation as

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega)(1+z)^2}\,,$$

where the density parameters are as they would be measured today and

$$\Omega = \Omega_r + \Omega_m + \Omega_\Lambda\,,$$

is the total energy density.

As we shall see in Section 4, the energy density today in radiation is small $\Omega_r \sim 10^{-5}$, while today $\Omega_m \sim \Omega_\Lambda$ are $\mathcal{O}(1)$. As the equation above makes clear, going back in time (large z) radiation was dominant. Primordial nucleosynthesis (Section 4) indicates that the expansion of the universe was initially radiation dominated (RD). As the energy density in radiation decreases more rapidly than that of

matter, the universe became matter dominated (MD) (after a time called matter-radiation equality) and then, it turns out, Λ dominated (LD).

The result is that mapping the Hubble parameter H as a function of redshift z would give direct information about the composition of the universe. The most straightforward way to do so would be to measure the physical distance of distant galaxies *today*. Consider again a light ray emitted at time t_e by some galaxy and observed today t_0. The time lapse is related to the physical position of the galaxy today by

$$d_P|_{\text{today}} \equiv \chi = \int_{t_e}^{t_0} \frac{dt}{a}\,.$$

Changing variables ($a = 1/(1+z)$) we get a beautiful relation

$$\chi(z_e) = \int_{a_e}^{1} \frac{da}{Ha^2} = \int_0^{z_e} \frac{dz}{H(z)}\,. \tag{7}$$

To have some hindsight, consider a flat, matter universe. Then you may check that

$$\chi(z_e) = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z_e}}\right).$$

At small redshifts $z \lesssim 1$ the relation is linear, as the one observed by Hubble Eq. (1),

$$H_0\chi \approx z_e\,.$$

All solutions reduce to this relation for small enough redshifts. This is only an approximation. As we shall see, the first correction is related to the deceleration or acceleration of the expansion.

At large redshifts, we get something interesting

$$\lim_{z_e\to\infty} \chi(z_e) = \frac{2}{H_0}\,.$$

This is the largest distance that a signal propagating at the speed of light could have travelled in (such a) universe. This limiting distance is called the *particle horizon*. Generically and at any time, the horizon is at a distance set by the Hubble parameter

$$d_{PH} \sim 1/H\,.$$

The relation (7) is nice, but unfortunately not yet tractable. The reason is that we do not have a direct access to the quantity χ. Remember that this is the distance to an object *today*. Such a distance may be established by bouncing light rays between near neighbours but, on cosmological distances this is, well, difficult to do. In practice what we measure is the light coming from distant galaxies. If we knew the absolute luminosity $\mathscr{L}$ of the emitting galaxy, then, *in a static universe*, its apparent luminosity $\mathscr{F}$, equal to the energy flux received per steradian, is given by energy conservation

$$\mathscr{F} = \frac{\mathscr{L}}{4\pi d_L^2}\,.$$

We can thus define the *luminous distance* by

$$d_L = \left(\frac{\mathscr{L}}{4\pi\mathscr{F}}\right)^{\frac{1}{2}}.$$

This definition may be applied to an expanding universe, provided we understand how the energy flux is affected by expansion.

Using comoving distance, in a flat, static universe the flux observed today is

$$\mathscr{F} = \frac{\mathscr{L}}{4\pi\chi^2}\,.$$

What about a curved, expanding universe? In curved space we must first replace χ by $S_k(\chi)$ (defined in Section 2) *i.e.*, the area of a sphere of radius χ is smaller (larger) in a spherical (respectively hyperbolic) universe, see Fig. 16. Furthermore, because of expansion the energy of a photon is redshifted by a factor of $a = 1/(1+z)$ between emission and reception. Last, we must take into account the fact that the rate of photon reception is smaller than the rate of emission by a factor of $a = 1/(1+z)$. Altogether, the flux

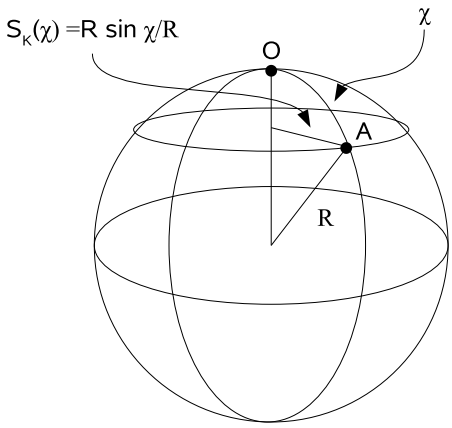


Fig. 16: On a sphere, the circumference of a circle of radius χ is given by $2\pi S_k(\chi)^2$.

of energy received is

$$\mathscr{F} = \frac{\mathscr{L}a^2}{4\pi S_k^2(\chi)}\,.$$

This motivates to define the *luminous distance* as

$$\boxed{d_L = \frac{S_k(\chi)}{a} = (1+z)S_k(\chi)\,.}$$

Note that, in a flat universe, this is a factor of $(1+z)$ larger than the physical distance today

$$d_L = d_P(1+z)\,.$$

Objects look fainter, *i.e.*, further away, because their light is redshifted by expansion.

Although we shall not use it before Section 7, we introduce here yet another useful definition of distance. Suppose we know the physical size D of an object. In Euclidean space, its apparent diameter δ is

$$\delta = \frac{D}{d_A}$$

if the object is at distance d_A. Thus we define

$$d_A = \frac{D}{\delta}$$

and ask how D and δ are affected in an expanding universe.

According to the FRW metric, the angle sustained by an object of physical size D at comoving distance χ is

$$\delta = \frac{D}{aS_k(\chi)},$$

which gives

$$\boxed{d_A = aS_k(\chi) = \frac{S_k(\chi)}{1+z}.}$$

Note that in a flat universe, the angular distance is *smaller* than the physical distance by a factor of $1/(1+z)$,

$$d_A = d_P/(1+z).$$

The three notions of distance d_P, d_L and d_A give the same answer for small redshifts but they depart from each others at large redshifts. For instance, if we remember that d_P is limited by the particle horizon, $d_P \to 2/H_0$, we see that the angular distance first increases, reaches a maximum, and then decreases with distance (see Fig. 17). The effect of curvature is also both interesting and important. In a

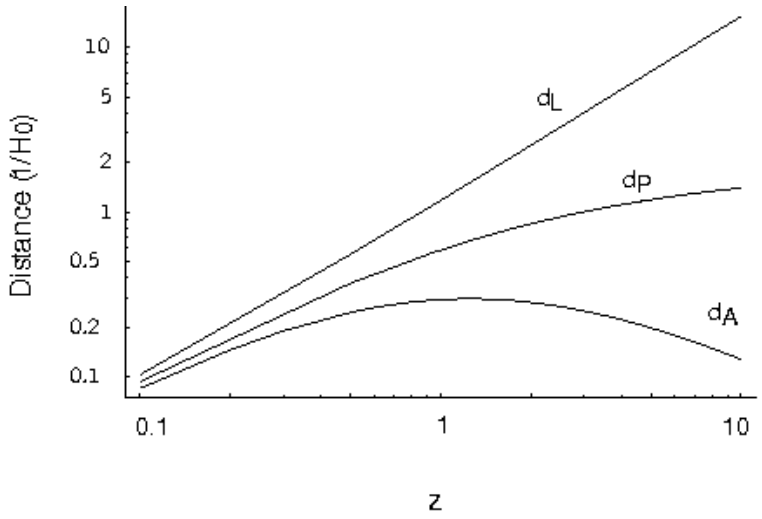


Fig. 17: Comparison of the three notions of distance discussed in the text for the case of a flat, matter-dominated universe.

curved universe,

$$d_A(z) = \frac{1}{(1+z)H_0\sqrt{|\Omega_k|}} \begin{cases} \sinh(\sqrt{\Omega_k}H_0\chi) & \Omega_k > 1 \\ \sin(\sqrt{-\Omega_k}H_0\chi) & \Omega_k < 1 \end{cases} \tag{8}$$

where $\Omega_k = 1 - \Omega = -K/H_0^2$. An object of fixed size, at fixed comoving distance, appears larger in a closed universe than in a flat universe. The converse holds in an open universe. Figure 18 compares the different distances for three universes.

We have seen that the luminous, physical and angular reduce to (1) at small z. Although we may plot (or derive analytically in some cases) the Hubble diagram for different models, it is useful and of interest to express the $\mathcal{O}(z^2)$ correction to the Hubble law in a model, independent way. We focus on the luminous distance and express d_L as a function of the redshift z so we first need to eliminate the reference to χ. This is easy if we can solve the Friedmann equation for $a(t)$, as we did in the figure in the preceding section. Otherwise, if we consider small z, we can derive an approximate relation by expanding $a(t)$ near $t = t_0$. This approach does not rest on theoretical prejudices (*i.e.* the validity of Einstein equations) but requires the introduction of more parameters.

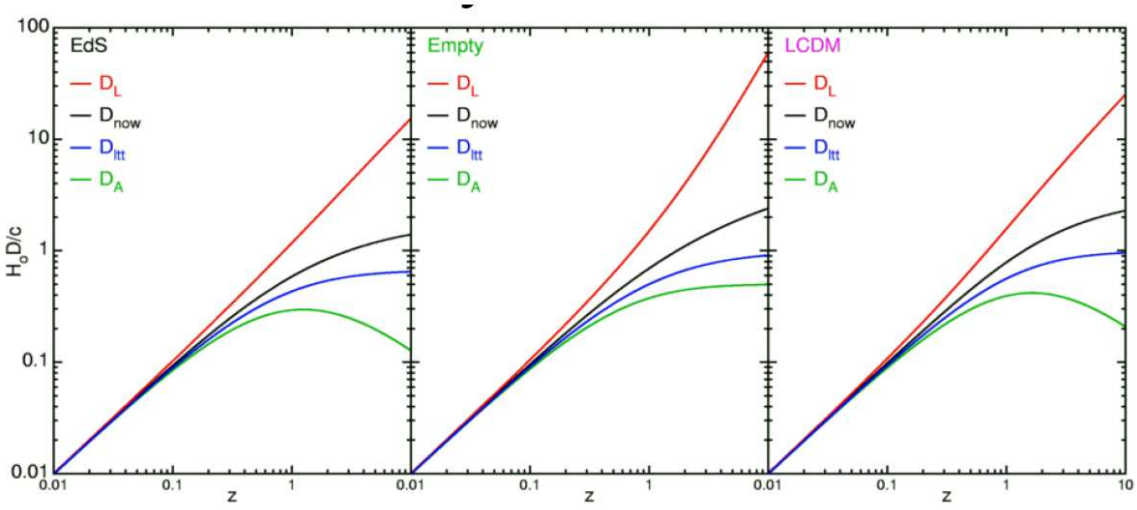


Fig. 18: Comparison between d_L, $d_{\rm now} \equiv \chi$ and d_A for the Einstein-de Sitter universe (flat, matter-dominated), an empty universe, and a flat, Λ plus matter universe. See Ned Wright's tutorial, from where the figure is taken [7]). All distances agree for small redshifts. Note that the presence of a cosmological constant make objects seem further away (dimmer) than if there is only matter. Also objects in an hyperbolic universe (like an empty universe) sustain a smaller angle (larger d_A) than in a flat universe and *a fortiori* in a spherical universe.

We first need χ as a function of time t. From the geodesic motion of a photon, we have

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \chi\,. \tag{9}$$

Expanding $a(t)$ in the vicinity of t_0 gives

$$\begin{aligned} a(t) &= a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2 + \ldots \\ &= 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \ldots \end{aligned} \tag{10}$$

where $H_0 = \dot{a}/a$ today and $q_0 = -\ddot{a}_0/H_0$ is called the *deceleration parameter*. Inserting the expansion of $a(t)$ in the LHS of Eq. (9), we get

$$\chi = (t_0 - t_1) + \frac{1}{2}H_0(t_0-t_1)^2 + \ldots$$

In a static universe, it takes a time $t_0 - t_1$ for light to travel a physical distance χ (remember comoving distance = physical distance today). This takes less time in an expanding universe since the source was closer, $a(t_1)\chi < \chi$. Finally we need to relate $t_0 - t_1$ and z. As

$$1 + z = \frac{a_0}{a_1} = \frac{1}{a(t_1)}\,,$$

we have

$$z = H_0(t_0-t_1) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0-t_1)^2 + \ldots$$

or

$$t_0 - t_1 = H_0^{-1}\left(z - (1+q_0/2)z^2 + \ldots\right)\,.$$

Now, if we limit the expansion to second order in z, you can verify that the correction due to spatial curvature is to next order. That is, we can take

$$S_k(\chi) \approx \chi\,.$$

Putting everything together, we find that the luminous distance to the source is related to redshift by

$$\boxed{d_L H_0 = z + \frac{1}{2}(1-q_0)^2 z^2 + \ldots}\,.$$

Note that there is deviation from the Hubble law even in a universe with vanishing deceleration (Milne universe). The deceleration parameter q_0 is directly related to the matter/energy content. Consider for instance a universe with matter and a cosmological constant. Using the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{1}{2}H^2\left(\frac{\rho_m}{\rho_c} + \frac{\rho_\Lambda}{\rho_c} - 3\frac{\rho_\Lambda}{\rho_c}\right)$$

we finally get

$$q_0 = -\left.\frac{\ddot{a}}{aH^2}\right|_{t_0} = \frac{1}{2}\left(\Omega_m - 2\Omega_\Lambda\right).$$

We need one last piece of information before we have a look at the data. Astronomers use magnitude to express distances. The relation between magnitude and to luminous distance is given by

$$m = M + 5\log_{10}\left(\frac{d_L}{10\,\mathrm{pc}}\right) + K(z)\,,$$

where M is the magnitude of an object as seen from a distance of 10 pc. The K correction factor takes into account the fact that instruments are sensitive not to the total luminosity but to some range of frequencies and that frequencies are shifting because of expansion.

To probe deviations from the linear approximation to the Hubble law, we need to observe objects at redshift $z \sim 1$. Furthermore we need a way to estimate their distance with some confidence. Such objects are called 'standard candles'. Hubble used cepheids, the first stellar objects that allowed one to measure distances beyond our galaxy. The recent breakthrough has been realized by the use of a category of supernovae explosion, called type Ia supernovae, to probe the expansion of the universe. Their extreme luminosity and an apparent universality in the shape of their light curve have promoted them to the rank of standard candles. This is not without problems, including the fact that we do not actually properly understand the physics of SnIa explosions or the fact that the SnIa explosions at large redshifts took place in a younger universe, with possible evolution effects on the explosion of supernovae. Nevertheless, experts seem to agree that they are good candles and that they give us a faithful mapping of the cosmological expansion. A good review of the use of SnIa in cosmology is [8].

Figures 19 and 20 show the historical data published at about the same time at the end of the 1990s by two independent teams, the Supernovae Search Team [9] and the Supernovae Cosmology Project [10]. Both sets of data concur with indicating that the expansion of the universe is accelerating, rather than decelerating. A recent compilation of observations is shown in Fig. 21, taken from Ned Wright's cosmology tutorial. Clearly the data are in favour of a flat universe, made up of 30% matter and 70% cosmological constant. Incidentally, the data also gives $H_0 = 71$ km$\cdot s^{-1}\cdot$Mpc^{-1}. Figure 22 shows that a non-zero cosmological constant is necessary to fit the data, a most remarkable, albeit puzzling, result. We shall come back to the cosmological constant or its sibling, dark energy, in Section 8.

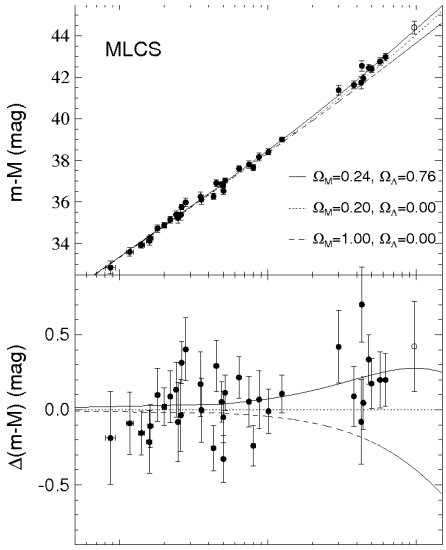


Fig. 19: Supernovae Hubble diagram from the Supernovae Search Team.

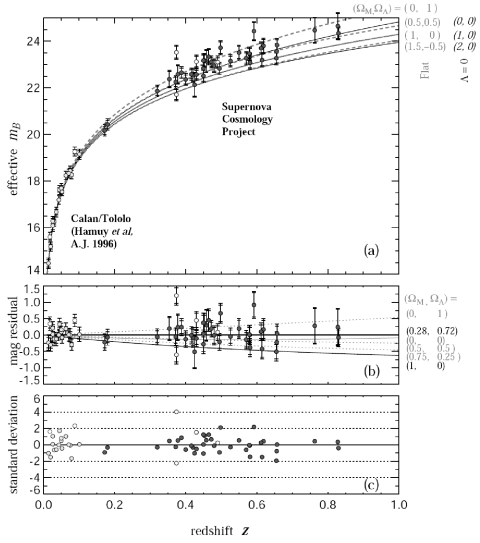


Fig. 20: Supernovae Hubble diagram from the Supernovae Cosmology Project.

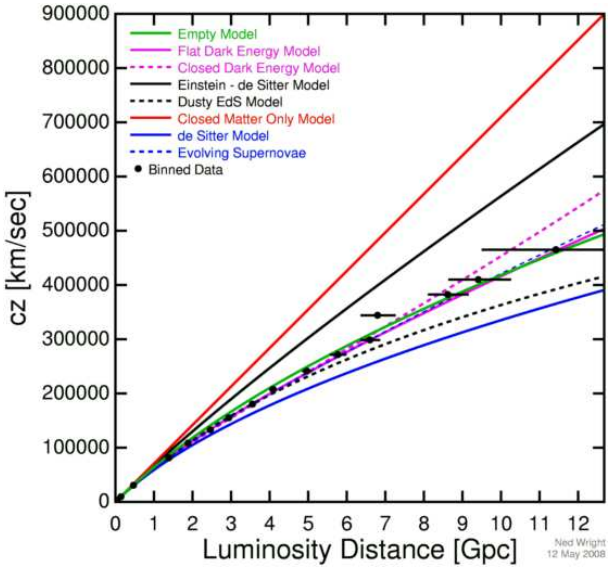


Fig. 21: Supernovae Hubble diagram confronted to different models of the universe. The best curve (in purple) is a flat universe with $\Omega_m = 0.27$ and $\Omega_\Lambda = 0.73$. From Ned Wright's cosmology tutorial.

4 The early universe

The cosmic microwave background radiation (CMBR) that fills the universe has the spectrum of a black body at temperature $T_0 = 2.725$ K. The corresponding number of modes in an interval of energy between ω and $\omega + d\omega$ is given by the Planck distribution

$$\mathcal{N}(\omega)d\omega = \frac{2}{\exp\omega/T - 1}\frac{d^3\mathbf{k}}{(2\pi)^3} \equiv \frac{\omega^2}{\exp\omega/T - 1}\frac{d\omega}{\pi^2}$$

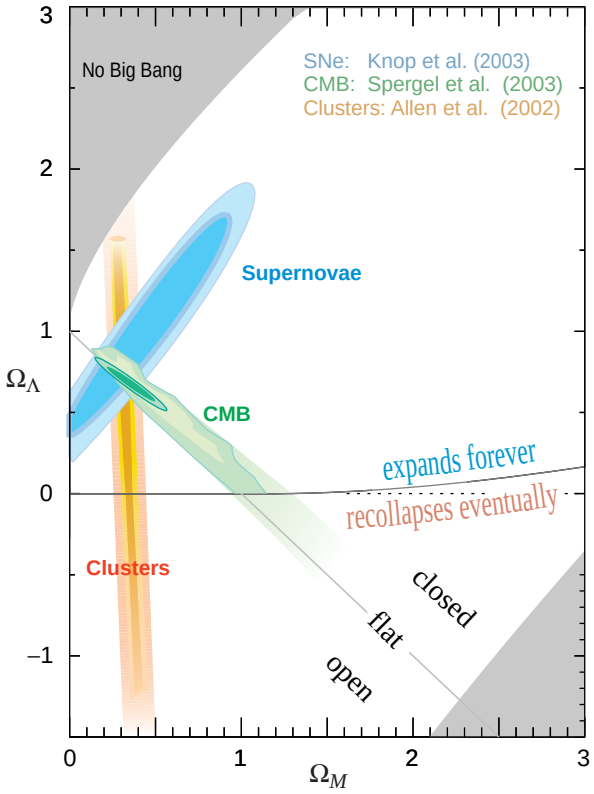


Fig. 22: Preferred regions in the $\Omega_m - -\Omega_\Lambda$ plane from supernovae data and results from observations of the CMB and large-scale structure and dynamics of clusters of galaxies, from the Particle Data Group [11].

with $\omega = k \equiv |\mathbf{k}|$, and we set $\hbar = c = k_B = 1$. The factor of 2 is the number of possible polarisation states of a photon. Integrating over energy gives the density of photons

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$$

with $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1.2020\ldots$ This is indeed a density, since $[T] = E = L^{-1}$ using natural units. Taking $T = T_0$ today, you may verify that there are about $4 \cdot 10^8$ photons per cubic metre. More precisely

$$n_{\gamma 0} = 410 (T_0/2.725)^3 \,\mathrm{cm}^{-3}\,.$$

As a way of comparison, suppose there is one galaxy like Andromeda per Mpc^3 on average. This would correspond to $\mathcal{O}(10)$ baryons per cubic metre. Later on we shall have better determinations of the number of baryons, but the conclusion will not change: there are many more photons than baryons in the universe.

The energy density in the CMBR is, however, quite small today compared to that in baryons. Using

$$\rho_\gamma = \int_0^\infty \omega \mathcal{N}(\omega) d\omega = \frac{\pi^2}{15} T^4\,,$$

one finds that today

$$\Omega_{\gamma 0} h^2 = 2.47 \cdot 10^{-5}\,.$$

One of the main goals of this section will be to get an estimate for Ω_{b0} and consequently the density of baryons n_{b0}. The important quantity $\eta = n_b/n_\gamma \ll 1$ is called the *baryon number* of the universe. For

the sake of reference, we adopt $\eta \approx 10^{-9}$. A more precise number will be given before the end of this section.

Given that $\rho_\gamma \propto T^4$ and $\rho_\gamma \propto a^{-4}$ in the expanding universe, we may conclude right-away that

$$T \propto a^{-1}$$

and also $n_\gamma \propto a^{-3}$, as expected.

It is perhaps worth emphasizing that the above results do not imply that photons of the CMBR are in thermal equilibrium *today*, for these photons have been travelling for billions of years and they have hardly been interacting with anything. However, it is direct proof that, once upon a time in the universe, light was in thermal equilibrium. This confidence rests on the neat fact that a thermal distribution of relativistic particles is preserved by expansion. Indeed, if photons do not interact then

$$a^3 \mathscr{N}(\omega) d\omega$$

should be constant as the photons can not be destroyed or created. Indeed, using $\omega \propto a^{-1}$ (remember that $\lambda \propto a$), we have

$$\mathscr{N} d\omega \propto \frac{\omega^2 d\omega}{\exp(\frac{\omega}{T}) - 1} \to \frac{1}{a^3} \frac{\omega^2 d\omega}{\exp(\frac{\omega}{a} \cdot \frac{a}{T}) - 1} \propto a^{-3},$$

where we have used $T \propto a^{-1}$. Hence photons cool down just as if they were in thermal equilibrium, even if they do not interact anymore with any thermal bath.

As we go back to the past, the mean energy of photons was larger by a factor of a^{-1}. Eventually this was enough energy for photon to ionize hydrogen, provided say $T \gtrsim$ few eV. At even higher temperature, pairs of electrons and positrons were constantly created and destroyed ($T \gtrsim m_e$), baryons were not in hadrons but existed as free quarks ($T \gtrsim 1$ GeV), etc. The more we go back in the past of the universe, the more particle species were relativistics and were in thermal equilibrium. Basically, we expect that, if the universe was ever has hot at $1TeV$, all particles of the Model Standard (corresponding to $\mathscr{O}(100)$ degrees of freedom) had an abundance

$$n \propto T^3$$

characteristic of a gas of relativistic particles.

Let $g_\star$ be the effective number of relativistic degrees of freedom at T, *i.e.* such that $m \lesssim T$. Then, following Kolb and Turner [12], it is convenient to write

$$\rho_r = g_\star \frac{\pi^2}{30} T^4$$

with

$$g_\star = \sum_{B_i=\text{bosons}} g_{B_i} + \frac{7}{8} \sum_{F_i=\text{fermions}} g_{F_i}.$$

The factor of $7/8$ for fermionic degrees of freedom is there because they obey Fermi-Dirac rather Bose-Einstein statistics

$$\mathscr{N}_F(\omega) d\omega = g_F \frac{\omega^2}{\exp \omega/T - 1} \frac{d\omega}{\pi^2}.$$

For instance $g_{e^-} = g_{e^+} = 2$ because of spin, while $g_{\text{up quark}} = 2 \times 3 = 6$ because they come in three colours. The number density for relativistic fermion species is

$$n_F = \frac{3}{5} \frac{\zeta(3)}{\pi^2} T^3.$$

Again the factor $3/5$ comes from the difference between the Fermi-Dirac *vs* Bose-Einstein statistics.

The expansion rate takes a very simple form in the radiation dominated era of the universe. Since $\rho_r \propto a^{-4}$, at early times radiation is bound to be more important than matter, curvature or a cosmological constant. The Friedmann equation then reduces to

$$\boxed{H \approx 1.66 g_\star^{1/2} \frac{T^2}{M_{Pl}}}$$

where we have introduced the Planck mass $M_{Pl} = \sqrt{\hbar c/G} \equiv 1.2209 \cdot 10^{19}$ GeV. In the early universe $a \propto t^{1/2}$ and thus $H = 1/2t$. The age of the universe when the temperature was T is thus

$$t = 0.30 g_\star^{-1/2} M_{Pl}/T^2$$

For instance $T = 1$ TeV at $t \sim 10^{12}\,\mathrm{GeV}^{-1} \sim 10^{-13}$ sec for $g_\star \sim 10^2$.

What about $T \sim 1$ MeV? At that temperature the only relativistic degrees of freedom were photons, electrons and positrons and the three flavours of neutrinos. The protons and neutrons were also present but they were non-relativistic and much less abondant than the relativistic species. Hence $g_\star = 2 + 7/8 \cdot (4 + 3 \times 2) = 10.75$ (supposing that only the L-helicity neutrinos were in thermal equilibrium).

Was the universe radiation or matter dominated at $T \sim 1$ MeV? We have already argued that cosmological data point to $\Omega_m = \mathcal{O}(1)$ today, while we know that $\Omega_\gamma \approx 5 \cdot 10^{-5}$ today. Consequently the energy density in matter was equal to that of radiation when $a_{EQ} = \Omega_r/\Omega_m \sim 10^{-4}$, corresponding to $T_{EQ} = T_0/a_{EQ} \sim 30,000$ K~ 3 eV. We shall have a more precise determination of the temperature and redshift at the time of *matter–radiation equality* when we know more about the density of matter and radiation in the universe. Suffices it to say that $T_{EQ} \sim 3$ eV is a good estimate and that at $T \sim 1$ MeV, the universe was definitively radiation dominated. You may now check that $t \sim 1$ s at $T \sim 1$ MeV while $t_{EQ} \sim 10^5$ years.

The temperature $T \sim 1$ MeV is of the order of the difference between the neutron and proton masses

$$Q = m_n - m_p = 1.293\,\mathrm{MeV}\,.$$

In thermal equilibrium, the relative abundance of neutrons and protons is Boltzmann suppressed

$$\frac{n_n}{n_p} = e^{-Q/T}\,. \tag{11}$$

This stems from

$$n_i = g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}$$

which is the Boltzmann-Maxwell distribution for a non-relativistic species of mass m and chemical potential μ. Taking the ratio of neutron and proton abundances, neglecting the chemical potentials[6] and the difference in mass in the prefactor gives (11).

At $T > 1$ MeV, there are as many neutrons and protons, $n_n \approx n_p$, as expected. As the temperature falls, however, protons become more prominent. However thermal equilibrium abundances may be maintained only if weak processes

$$n + \nu_e \leftrightarrow p + e$$

or

$$n + \bar{e} \leftrightarrow p + \bar{\nu}_e$$

[6]The chemical potentials in (11) drop for the following reason. In chemical equilibrium, the chemical potential satisfies $\mu_n - \mu_p \equiv \mu_e - \mu_\nu$. Since the universe is neutral $\mu_e/T \equiv \mu_p/T \ll 1$, where the latter is because there are many more photons than protons. The chemical of the neutrinos is not known but provided there is no large asymmetry between neutrinos and antineutrinos μ_ν/T should be small too. See for instance Kolb and Turner.

are efficient. The rate Γ of these processes is controlled by a cross-section typical of weak interactions. In a thermal bath the cross-section is

$$\langle \sigma |v| \rangle \sim G_F^2 T^2$$

where $G_F \approx 10^{-5}$ GeV^{-2} is the Fermi constant (you can guess this purely on dimensional grounds, replacing energy by temperature) while the density of target particles $n \sim T^3$. Thus the rate is typically

$$\Gamma \sim G_F^2 T^5 .$$

This interaction rate has to be compared with the expansion rate of the universe $H \sim g_*^{1/2} T^2 / M_{Pl}$. A process is efficient in maintaining thermal equilibrium iff

$$\boxed{\Gamma \gtrsim H .}$$

Otherwise, it is said to be out of equilibrium. Intuitively, departure from equilibrium happens when particles are taken apart by expansion faster than they can interact.

Concretely, the weak processes in which the neutrons and protons take part become inefficient (we will say that interactions freeze out) when

$$\Gamma \sim H \ \rightarrow \ G_F^2 T^5 \sim g_\star^{1/2} \frac{T^2}{M_{pl}} .$$

The freeze out (F0) temperature is

$$T_{FO} \sim 1\,\mathrm{MeV} .$$

The estimate above is crude but it captures the essence of the physics. A more precise calculation would give $T_{FO} \approx 0.8$ MeV.

The bottom line is that below $T_{FO} \approx 0.8$ MeV, neutrinos stop interacting. Free neutrons may still decay (the mean lifetime of a neutron is about 900 seconds) but protons may not be transformed back into neutrons.

At T_{FO}

$$\frac{n_n}{n_p} = e^{-Q/T_{FO}} \approx 1/5 .$$

Neutrons either decay or are bound in nuclei. Around T_{FO}, processes like

$$p + n \leftrightarrow D + \gamma$$

or

$$D + D \leftrightarrow {}^4He + \gamma$$

are very efficients and thus in equilibrium. These are strong interaction processes, while processes with neutrinos are weak interactions, so let us assume as a first guess that all the neutrons are rapidly bound in helium nuclei after freeze-out. The mass fraction of ^{4}He (how much baryon mass is in helium) would then be

$$X_{He} = \frac{m_{He} n_{He}}{m_n n + m_p p} \approx \frac{4 n_{He}}{n + p} = \frac{2n}{n + p} \approx \frac{1}{3}$$

where we have used the fact that they are two neutrons per helium nuclei. For a first estimate, this is not too bad since the observed[7] mass fraction of primordial ^4He is close to 25%. In the same approximation, the left-over protons will eventually bind with electrons to form hydrogen, the most abundant form of

[7] Since helium is also produced in stars, it is a complicated matter to relate the observed abundance to the primordial one. We do not have time to cover this subject here but Kolb and Turner or the fairly recent review [13] are places to start.

ordinary matter in the universe. This program is called *primordial nucleosynthesis*, to differentiate it from the nucleosynthesis that will take place in stars (and explosions of stars) much later in the history of the universe. Although helium is also created in stars (like in the Sun), it can account for only a small fraction of all the helium seen in the universe. The necessity of primordial nucleosynthesis is thus quite well established. It is actually one of the three pillars of the Big Bang model, together with the recession of galaxies and the thermal character of the CMBR.

The explanation for the difference between our estimate and observations is interesting. The first thing is that helium abundance is very small at freeze-out and that nucleosynthesis takes place much later on, around $t \approx 3$ min. Between $t \approx 1$ s and $t \approx 3$ min, neutrons have time to decay substantially so that

$$\left.\frac{n_n}{n_p}\right|_{FO} \approx \frac{1}{5} \longrightarrow \left.\frac{n_n}{n_p}\right|_{t\approx 3\text{min.}} \approx \frac{1}{7}$$

which gives, assuming again that most neutrons go into helium nuclei,

$$X_4 \approx 0.25\,.$$

This is what is observed (see however the previous footnote).

We still have to understand why the abundance of helium is small at $t \approx 1$, $T \sim 1$ MeV? The binding energy of ^{4}He is $B_4 = 28.3$ MeV so on energy grounds we would expect all neutrons to be in bound states. One way to make ^{4}He is through deuterium D, an isotope of hydrogen with one neutron. So to make ^{4}He we have to make sure that there is D. The binding energy of deuterium is $B_2 = m_n + m_p - m_D = 2.22$ MeV, also larger than $T \sim 1$ MeV so we expect D to be abundant too, and thus helium to form, etc. But this is not the case because there are many more photons than baryons in the universe $\eta = n_b/n_\gamma \approx 10^{-9}$. These many photons may efficiently dissociate nuclei. The result is that in equilibrium

$$X_2|_{Eq.} \sim \eta e^{B_2/T}$$

while, for nuclei made of A nucleons

$$X_A \sim \eta^{A-1} e^{B_A/T}\,.$$

These so-called Saha equations [12] tell that there is a competition between energy (the tendency to make bound states $\propto e^{B/T}$) and entropy (the many ways they may be dissociated $\propto \eta^A$). At $T_{FO} \sim 1$ MeV,

$$X_2 \approx 10^{-12} \qquad X_4 \approx 10^{-23} \qquad X_{12} \approx 10^{-108}\,.$$

These abundances are very small and the conclusion is that the temperature has to drop before nucleosynthesis may really begin. Solutions of the Boltzmann equations for the abundance of light nuclei are shown in Fig. 23. An important feature is that not all D is burnt into ^{4}He and there is a relic abundance of deuterium. This abundance turns out to be very sensitive to the baryon number η, as Fig. 24 reveals. The effect on ^{4}He is easy to understand. If there are fewer photons (larger η), nucleosynthesis would start earlier (higher temperature, thus more neutrons would be left) and X_4 would be larger.

The comparison of observations to prediction of primordial nucleosynthesis gives

$$\eta = (6.0 \pm 0.15) \cdot 10^{-10}$$

or

$$\Omega_b h^2 = 0.020 \pm 0.05\,.$$

This is a remarkable result. If we believe in primordial nucleosynthesis (you should!), then we know how many baryons there are in the universe, even though we can not see most of them (most baryonic matters is in the interstellar medium, not in stars)!

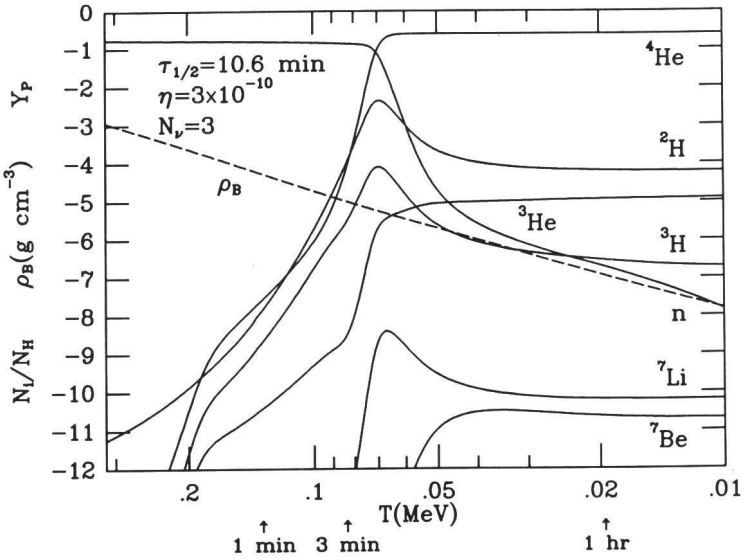


Fig. 23: Typical solutions of the Boltzmann equations describing the evolution of the abundance of light nuclei in the early universe. Most neutrons go into ^{4}He but there are also relic abundances of D (^{2}H), tritium (^{3}H) and lithium and beryllium. From Kolb and Turner.

The abundance of ^{4}He is also sensitive to the number of relativistic degrees of freedom at $T \sim 1$ MeV. For instance, if there were more light neutrinos, then the expansion of the universe at $T \sim 1$ MeV would be larger, which would lead to freeze-out at a higher temperature, thus more remnant neutrons, and thus more ^{4}He would be formed. The current limit from primordial nucleosynthesis on the number of light ($m \lesssim 1$ MeV) neutrino families is

$$1.8 < N_\nu < 4.5 \qquad \text{(PDG)}\,.$$

This limit predates (and is consistent with) the limit on the number of neutrino families from measurement of the width of the Z at LEP1. This is a neat example of the interplay between cosmology and high-energy physics.

Primordial nucleosynthesis implies that baryonic matter represent only 5% of the critical energy density. In the previous section we have seen that the contribution of all matter should be about 30%. This means that, on top of baryonic or ordinary matter, there should be another form of matter in the universe. This is called dark matter.

5 Dark matter

Most matter in the universe is not visible and indications that this matter is not made of baryons are strong. With increasing level of confidence, these are

1. The spiral galaxies rotation curve problem

Plots of the orbital velocity of stars and of the interstellar gas in spiral galaxies (in particular the so-called HI regions, a halo of ordinary matter which extends beyond the distribution of stars in spiral galaxies and is composed of neutral atomic hydrogen, visible through 21 cm emission) are in discrepancy with a naive application of Newton's law according to which

$$\langle v^2 \rangle \sim \frac{GM(r_<)}{r}$$

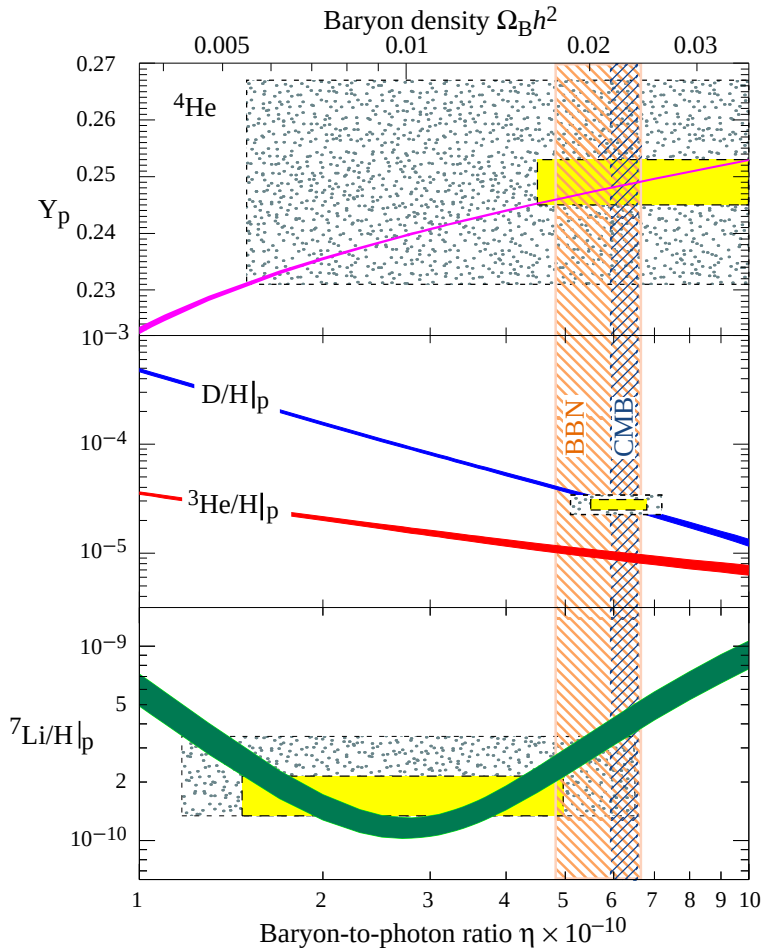


Fig. 24: Predictions of the Big Bang model for the primordial abundances (mass fractions) of light elements confronted to observations (boxes). Small boxes given 2σ statistical errors, the big boxes include systematic errors [11].

where $M(r_<)$ is the mass within radius r from the centre of the galaxy. Far from where most of visible matter is observed ($r >$ a few kpc) one expects $v \propto r^{-1/2}$. What is observed instead (on average, see Fig. 25) is roughly a plateau with $v \propto$ constant. One possible interpretation is that there is a halo of matter (composed of non or weakly interacting, non-relativistic objects) with $M(r_<) \propto r$ or $\rho \propto r^{-2}$.

The possibility that this halo is dominantly composed of massive astrophysical compact halo objects (MACHOs), like small stars (brown dwarfs), large planets (jupiters) or black hole is excluded for masses $10^{-7}\mathrm{M}_\odot < m <$ few $\mathrm{M}_\odot$ by observations made in the 1990s (EROS and MACHO collaborations).

The spiral galaxies rotation curve problem is also the main motivation for the MOND proposal (for Modified Newtonian Dynamics), an empirical modification of the laws of dynamics which is able to explain the shape of velocity curves without recourse to the existence of extra matter. This proposal is, however, challenged by observations made on the scale of clusters of galaxies, in particular the so-called 'Bullet cluster'.

To conclude, we add that the distribution of dark matter in the galaxy (if any) is not well known. It is expected to be more clustered at the centre of galaxies, where visible matter is also more

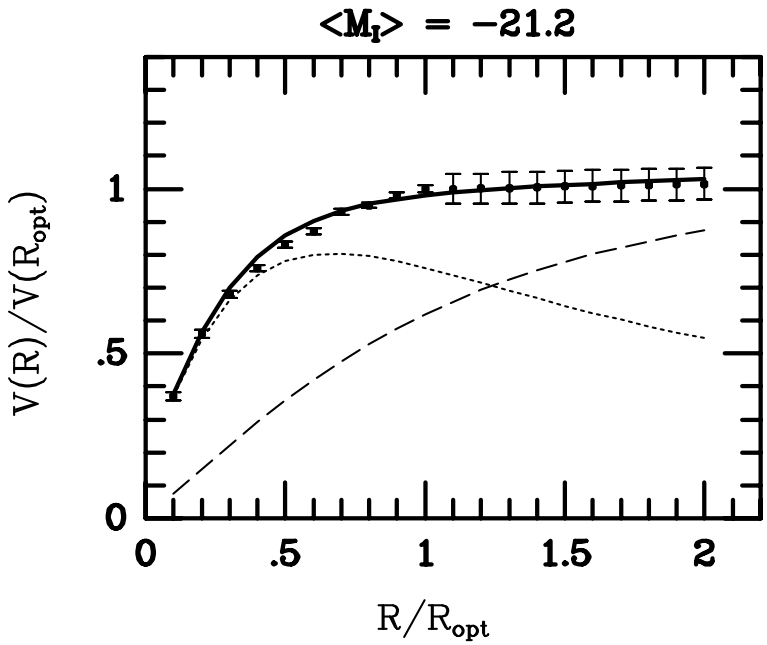


Fig. 25: Orbital velocity curve on average based on a sample of spiral galaxies (see Ref. [14] for details). The dotted curve is the expected behaviour based on the distribution of ordinary matter. The dashed curved is the contribution of hypothetical distribution of dark matter.

concentrated but there is no consensus yet.

For reference, to explain the rotation curve in our galaxy, we need on average $\rho_{dm} \approx 0.3\,\mathrm{GeV}\cdot\mathrm{cm}^{-3}$ at the position of the Solar system (about 8 kpc from the centre of the galaxy).

2. Clusters of galaxies

The dynamics of galaxies in clusters is historically the first hint for the existence of invisible matter. Studying the Coma cluster in 1933, Fritz Zwicky[8] showed using the virial theorem that the velocity of individual galaxies was too large to explain this system of galaxies as a relaxed, bound system, unless more, invisible matter is present. The amount of invisible matter is measured by the mass-to-light ratio M/L ($M/L = 1$ for the Sun), with $M/L \sim 100$ for clusters of galaxies.

A recent and most convincing indication for the presence of dark matter in clusters is the so-called Bullet cluster, a system which consists of two colliding clusters of galaxies. Matter in the Bullet cluster has been studied in the visible (which gives the distribution of galaxies), through gravitational lensing (which probes the shape of the Newtonian potential) and X-rays (which probes the presence of inter-galactic hot gas). Figure 26 is a composite showing all three components. In this figure there is a clear offset of the centre of mass of the two clusters.

The lore is that a cluster of galaxies is composed, with increasing importance in mass, of galaxies, inter-galactic gas (*i.e.*, the majority of ordinary matter), and dark matter. The interpretation of the figure is that, as the clusters passed through each other, both galaxies and dark matter went through while the inter-galactic gas, which has electromagnetic interactions, slowed down through collisions, forming the arrow-shaped shock front.

3. Large-scale structure

The most reliable indication for dark matter is the large-scale structure of the universe. Explaining this will be the main topic of of Section 7. Suffice it to say here that our confidence rests on the fact that, because inhomogeneities were small initially, the physics underlying the early formation

[8] Search for "spherical bastard" on the web.

Fig. 26: The Bullet cluster is actually two clusters in collision. This image is a composite showing the distribution of mass (in blue from gravitational lensing), that of the inter-galactic gas (the dominant form of ordinary matter in a cluster, in red from X-ray imaging), and that of galaxies, whose centre of mass coincides with that of the blue regions.

of the large-scale structure of the universe may be studied in linear approximation (while galaxies and clusters of galaxies are very complex, non-linear structures). Observations (anisotropies of the CMB and large-scale surveys, as those discussed in the first section) indicate there is about 5 times more dark matter than ordinary or baryonic matter. Moreover, this dark matter is likely to be composed of non-relativistic or mildly non-relativistic particles. This will imply that the neutrinos of the Standard Model can not be the dominant form of dark matter.

The particle answer to the dark matter problem is that dark matter is composed of particles. After all, there is dark matter within the Standard Model itself. Indeed massive neutrinos are dark matter candidates, since they interact so weakly with baryons and light and they are abundant in the universe. However, neutrinos are too light to be the dominant component of dark matter.

Tritium decay puts the limit $m_\nu < 2$ eV while solar and atmospheric oscillations for three families constrain the mass difference (squared) between neutrino generation, giving respectively

$$\Delta m^2_{21} = (8.0 \pm 0.3) \times 10^{-5}\,\mathrm{eV}^2$$

and

$$\Delta m^2_{32} = 1.9 \,\mathrm{to}\; 3.0 \times 10^{-3}\,\mathrm{eV}^2\,.$$

These bounds $m_\nu \lesssim 1$ eV imply that neutrinos are instances of something called Hot Dark Matter (HDM) a form of dark matter not consistent with large-scale structures (see Section 7). This is consistent, however, with the standard lore according to which, given $m_\nu \lesssim 2$ eV, neutrinos in the universe are too few to be the dominant form of dark matter. The argument goes as follows.

According to the discussion of Section 4, neutrinos decoupled in the universe at a temperature $T_{FO} \sim 1$ MeV. If we assume that the leptonic asymmetry was small (say, as small as the baryon asymmetry), at freeze-out, the abundance of each species of neutrinos was

$$n_{\nu_i} = \frac{3}{4} n_\gamma \propto T^3\,.$$

After freeze-out, the total number of neutrinos can not change anymore. Assuming that the total number of photons also stayed constant (see later), we would get $n_\nu \approx 308\ \mathrm{cm}^{-3}$ neutrinos per species today or $\rho_{\nu_i} \approx 3.3 \cdot 10^{-30} (\sum_i m_{\nu_i}/6\,\mathrm{eV})\, g{\cdot}\mathrm{cm}^{-3}$. Taking the limit from tritium decay gives an upper bound

$$\Omega_\nu \lesssim 0.18\,.$$

This is less (but not much,) than the abundance of dark matter $\Omega_{dm} \approx 0.25$. However, there is a subtlety: we have assumed that the number of photons is conserved but short after freeze-out positrons and electrons became non-relativistic and annihilated each other. Doing so, their entropy got transfer into photon entropy (*i.e.*, electron-positron annihilations adding more photons to the thermal bath). The net result is that the abundance of neutrinos is suppressed with respect to that of photons by a factor of $4/11 \approx 1/3$ (see Kolb and Turner if you want to know how get to this factor). Taking this suppression factor into account finally gives

$$\boxed{\Omega_{\nu,0} h^2 = \frac{\sum_{\nu_i} m_{\nu_i}}{94\,\mathrm{eV}}\,.}$$

The limit on the mass of neutrinos that we may get from this result is called the Cowsik-McLellan bound. Taking the experimental constraints gives

$$5 \cdot 10^{-4} \lesssim \Omega_{\nu,0} h^2 \lesssim 6.4 \cdot 10^{-2}\,.$$

Cosmic neutrinos and the CMBR photons are instances of relics from the early universe. The Standard Model of particle physics being not a complete theory, we may speculate about the existence of other relics. For instance, supersymmetric extensions of the SM require the existence of about a hundred new particles. If the early universe was hot enough, these particles were also in thermal equilibrium. The supersymmetric partners of SM particles are supposed to be odd under a discrete symmetry, called R-parity. The lightest supersymmetric particle (LSP) is then predicted to be stable (there are variations on this scenario). If it is neutral (thus at most weakly interacting, like the SM neutrinos), it could be a dark matter candidate. This idea is particularly appealing because the relic abundance of a non-relativistic particle with weak interactions (a weakly interacting massive particle or WIMP) is expected to be $\Omega_{WIMP} = \mathcal{O}(1)$ as we shall now see.

The WIMP scenario is not specific to supersymmetry, so consider a generic albeit hypothetical massive, stable, neutral and weakly interacting particle noted X. We suppose that X was in thermal equilibrium in the early universe. At high temperatures $T \gtrsim M_X$, the abundance was like that of photons, $n_X \sim T^3$ but when it became non-relativistic $T \sim M_X$, it was Boltzmann suppressed

$$n_X = g_X \left(\frac{M_X T}{2\pi}\right)^{3/2} e^{-M_X/T}\,.$$

We assume that there is no asymmetry in the abundance between X and $\bar{X}$ (no chemical potential) or, as in the supersymmetric scenario, that X is a real particle in which case $\bar{X} \equiv X$ (real scalar or Majorana fermion).

From our discussion of primordial nucleosynthesis, we saw that thermal equilibrium is maintained as long as X interactions are fast with respect to the expansion rate of the universe. In particular, the abundance of X is controlled by its annihilation into other particles (typically Standard Model in most scenarios, but there are variations here)

$$X + \bar{X} \leftrightarrow y + z$$

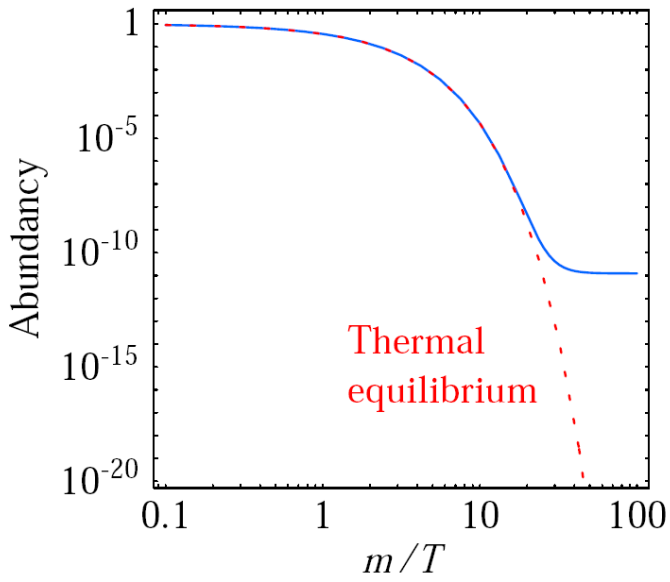


Fig. 27: Evolution of the abundance n_X/T^3 of a particle X whose interactions freeze out while it is non-relativistic. Increasing time is towards the right. The higher the interaction rate, the smaller the relic abundance.

If the cross-section (thermally average because we are in a thermal bath and the participating particles have a distribution rather than a precise energy) is σ, the annihilation rate is given by

$$\Gamma = \langle \sigma |v| \rangle n_X$$

As $n_X \propto e^{-M_X/T}$ drops rapidly, the rate of annihilation may rapidly become smaller than the expansion rate and annihilations essentially stop. To determine precisely the relic abundance of X particles we should write (and solve) a few Boltzmann equations. Much intuition may be gained by using the rule of thumb that equilibrium is maintained as long as

$$\Gamma \gtrsim H$$

where H is the expansion rate. Thus freeze-out occurs at a temperature such that

$$\langle \sigma |v| \rangle n_X(T_{FO}) \sim g_\star^{1/2} \frac{T_{FO}^2}{M_{pl}}$$

which gives a relic abundance of

$$n_X|_{FO} \sim g_\star^{1/2} \frac{T_{FO}^2}{\langle \sigma |v| \rangle M_{pl}}$$

or today

$$\rho_{X0} = M_X n_X|_{\text{today}} \sim g_\star^{1/2} \frac{x_{FO}}{\langle \sigma |v| \rangle M_{pl}} T_0^3$$

where $x = m_X/T$ which, for weakly interacting particles $m_X/T_{FO} = \mathcal{O}(10,20)$. We have also used $n_X \propto a^{-3} \propto T^3$ after freeze-out. This is a beautiful relation: the abundance is simply inversely proportional to the annihilation cross-section. This makes sense since the higher the annihilation rate, the smaller the relic abundance. The typical evolution of the abundance is depicted in Fig. 27. You may check that

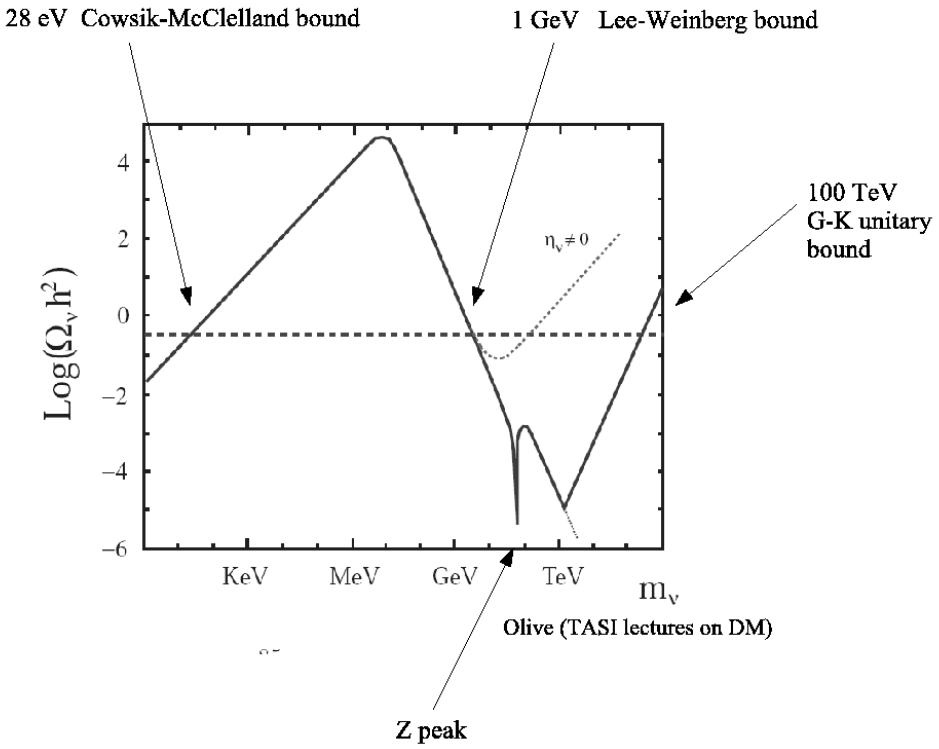


Fig. 28: Predictions of Ω_{dm} for a stable neutrino with SM interactions. At small masses $m_\nu \lesssim 1$ MeV, the neutrino interactions freeze out while it is relativistic (like the SM neutrinos). The Cowsik-McLelland bound gives $m_\nu \sim 30$ eV to get Ω_{dm} measured by WMAP. At higher masses, the interactions freeze out while the neutrinos is non-relativistic. Since $\sigma \propto G_F^2 E^2 \sim G_F^2 T^2$ at low energies, the cross-section increases with temperature and thus the relic abundance decreases. The peak is the Z resonance. At higher energies $m_\nu > m_Z$, the neutrino cross-section keeps increasing. This is specific to heavy neutrinos with SM-like interactions. For very large masses, however, unitarity requires that $\sigma \propto 1/m_\nu^2$ and the cross-section must decrease (this is beyond the SM, because then the neutrino must be strongly interacting but this is another story). The upper limit on the mass of a very massive neutrino is called the Griest-Kamionkowski bound. Finally, the dashed line shows the relic abundance if there was an excess of neutrinos over antineutrinos (non-zero neutrino chemical potential). The picture is taken from the review by K. Olive[15].

agreement with the observed abundance $\Omega_{dm} \sim 0.25$ requires $\sigma \sim 1$ pbarn, a cross-section typical of weak interactions.

This result is essentially independent of the mass of the dark matter candidate X. Given the interactions of X, we may thus have different possible X masses that are compatible with the dark matter abundance observed in the universe. This is illustrated in Fig. 28 for the case where X is a stable neutrino with the same interactions as the SM neutrinos.

The WIMP scenario explained here is quite generic and applies to many scenarios beyond the SM with new, stable particles. The most important feature (beside the appeal of a weakly interacting particle 'automatically' having the right abundance) is that the dark matter particle is typically heavy and belongs to a category called Cold Dark Matter (CDM). The relevance of this type of dark matter for the formation of large-scale structures will be shown in Section 7. First we would like to finish our survey of matter

with baryons. We believe that baryons are also relics of the early universe. However, the story is more involved than for dark matter.

6 Baryogenesis

There is much more matter than antimatter in the universe. There is essentially no antimatter on Earth. There is some antimatter in cosmic rays (at a level 10^{-5} antiprotons *vs* protons) but these are secondaries, produced in collisions. Also, if there were anti-galaxies, there would also be anti-galaxy/galaxy collisions with spectacular (!) productions of γ-rays and this is not observed.

We called the parameter $\eta = n_b/n_\gamma \approx 6 \cdot 10^{-10}$ the baryon number of the universe. It tells us that there are much more baryons than photons. This means that the asymmetry between baryons and antibaryons is very small. Indeed, consider the related quantity called the baryon asymmetry of the universe

$$\frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s}$$

where s is the entropy density

$$s = \frac{\rho + p}{T} \propto T^3 .$$

We have mentioned that the expansion of an isotropic and homogeneous universe is adiabatic or, in other word, isentropic. This means that sa^3 is a conserved quantity. We have also alluded to the concept of entropy when we discussed the relic density of neutrinos and the transfer of entropy from relativistic electron-positron pairs to photons. Most of entropy today is in photons but at any given temperature T it is shared among all relavistic species i such that $m_i \lesssim T$. The result is that the n_γ today is a measure of s. Also, if the baryon number is conserved, baryons and antibaryons may annihilate, but $n_B a^3$ stays constant. Hence

$$\frac{n_B}{s} = \text{constant} = \frac{n_b - n_{\bar{b}}}{s} \approx \left.\frac{n_b}{n_\gamma}\right|_{\text{today}} .$$

Hence, when $T \gg 1$ GeV, there were many baryons and antibaryons, with a little relative excess of the former, $\mathcal{O}(10^{-10})$.

We believe that the early universe was baryon symmetric, $n_B = 0$. One reason is aesthetic. A better one is that we believe that baryon number is actually not conserved by fundamental interactions. Baryon number violating processes in equilibrium in the early universe would then wash-out any pre-existing asymmetry. Yet another one is inflation. If there was an initial baryon asymmetry, it has been diluted by the exponential growth of the size of the universe during inflation.

So suppose there were as many baryons as antibaryons initially. The lighest baryons are protons and neutrons and they would annihilate with their antiparticles at $T \sim 1$ GeV, with a cross-section characteristic of strong processes $\sigma \sim 1/m_\pi^2$. If at that time there was no baryon excess, the relic abundance of baryons (and antibaryons) would be like in our discussion of dark matter,

$$n_b \sim n_{\bar{b}} \sim \frac{1}{\sigma}$$

Calculations give $T_{FO} \sim 20$ MeV (note that $x_{FO} = m_p/T_{FO} \sim 50$, larger than for weak interactions) and a residual abundance

$$n_b/n_\gamma \sim 10^{-20} .$$

This is called the 'annihilation catastrophe'. Basically it means that we need an excess of baryons before freeze out or otherwise there would be essentially no baryons left today.

So we must go from $n_B = 0$ early on to $n_B \neq 0$ before T_{FO}. This is called *baryogenesis*, a scenario first proposed by Sakharov in 1967. Baryogenesis in its simplest form necessitates three conditions to be realized.

1. that baryon number is not conserved
2. that the C and CP symmetries are violated
3. that there was departure from thermal equilibrium

Note that the last two conditions taken together amount to some form of CPT violation, the symmetry that relates particle to antiparticle properties.

Baryon number non-conservation is obviously mandatory to generate a baryon asymmetry from a symmetric initial condition. All know processes conserve the quantum number called baryon number, *e.g.*

$$n \to p + e + \bar{\nu} \tag{12}$$

has $\Delta B = 0$. It also has $\Delta L = 0$, conservation of the number of leptons. The lightest baryon is the proton and processes like

$$p \to \pi^0 + \bar{e}$$

which has $\Delta B = \Delta L = -1$ have never been observed. The current limit on this decay channel gives

$$\tau_p > 1.6 \cdot 10^{33} \,\text{years}\,,$$

much longer than the age of the universe $t_0 \approx 13 \cdot 10^9$ years.

The baryon number is one of the $U(1)$ global symmetries of the SM Lagrangian

$$\psi \to e^{iQ_B\alpha}\psi$$

and

$$\bar{\psi} \to e^{-iQ_B\alpha}\bar{\psi}$$

with $Q_B = 1/3$ for quarks and $Q_B = 0$ for all the other SM particles. We believe that global $U(1)$ symmetries are either accidental or remnant of a spontaneously broken local symmetry. Accidental symmetries are not protected. In the SM, $B+L$ is such an accidental symmetry[9]. It is conserved at the classical level of the theory, but it is broken explicitly at the quantum level. This is called a quantum anomaly. Thus there may be processes that violate $B+L$ *within* the SM. This turns out to be a quite subtle matter which entails the topology of the SM and instantons effects. The net result is that baryon number violating processes are very suppressed in vacuum. However, it is understood that baryon number violating processes may be efficient at high temperatures. This change is related to the effective restoration of the $SU(2)\otimes U(1)$ symmetry at high temperatures $T \gtrsim T_c \sim 1$ TeV.

That $B+L$ is not sacred is manifest in another aspect of physics beyond the SM. In grand unified theories (GUT), both baryons and leptons are in the same multiplet and thus may transform into each other. For instance in $SU(5)$, the simplest GUT, the processes of (12) predicted to occur with a rate

$$\Gamma \sim \frac{\alpha^2}{M^4} m_p^5$$

where $M \gtrsim 10^{16}$ GeV is the mass of the heavy gauge bosons in $SU(5)$. This scheme is not favoured by observations (unification of couplings gives too large a rate for baryon number violation) but the supersymmetric version of $SU(5)$ is still alive and well.

The baryon number changes under C and CP, and so a state with zero baryon number is an eigenstate of C, CP. If these symmetries are exact

$$[C(CP), H] = 0$$

[9] $B-L$ on the other hand is not broken by quantum effect. Incidentally it is a gauge symmetry in many extensions of the SM, like $SO(10)$.

and then

$$\langle B \rangle(t) = 0$$

for all time t.

Weak interactions break maximally C and P (the left and right chirality states have different interactions). CP violation is a more subtle effect. Within the SM it occurs through complex Yukawa couplings of the quarks, an effect which manifests itself in the phase in the CKM matrix (for three generations of quarks). Experimentally CP violation has been observed in the decay of K and B mesons. For instance

$$\frac{\Gamma(K_L \to l^+ \nu \pi^-) - \Gamma(K_L \to l^- \bar{\nu} \pi^+)}{\Gamma(K_L \to l^+ \nu \pi^-) + \Gamma(K_L \to l^- \bar{\nu} \pi^+)} = (3.27 \pm 0.12) 10^{-3} .$$

Hence it is likely that all the necessary ingredients for baryogenesis exist in nature. For the sake of argument, it is useful to consider a toy model of physics beyond the SM. Consider a heavy particle Y with may decay into SM particles (this could be on the heavy gauge bosons in $SU(5)$ alluded to above). Imagine there are two decay modes, with distinct baryon numbers (B is not conserved) and branching ratios r and $1-r$.

$$\begin{array}{lcll} X & \to & B_1 & r \\ X & \to & B_2 & 1-r . \end{array}$$

Then C and CP violation permit[10]

$$\begin{array}{lcll} \bar{Y} & \to & -B_1 & \bar{r} \\ \bar{Y} & \to & -B_2 & 1-\bar{r} \end{array}$$

with $r \neq \bar{r}$. Take a pair of Y and $\bar{Y}$. Their decay produces on average a baryon asymmetry

$$B_Y = rB_1 + (1-r)B_2 - \bar{r}B_1 - (1-\bar{r})B_2 = (r-\bar{r})(B_1 - B_2) .$$

If C, CP are conserved, $r = \bar{r}$ and there is no asymmetry. Idem if $B_1 = B_2$ of course. Why do we need both C and CP violation to get an asymmetry? Imagine that C and P are broken but that CP is conserved. Then, as Fig. 29 suggests, $r = \bar{r}$. If the conditions above are met, the decay of $Y - \bar{Y}$ pairs may produce an excess of baryons over antibaryons. However, in thermal equilibrium, processes that transform back baryons and antibaryons into Y and $\bar{Y}$ are also effective. The net effect is that no baryon asymmetry is produced. That a departure from thermal equilibrium is requisite may be understood on very general grounds. In thermal equilibrium the distribution of baryons and antibaryons are given

$$f_b(k) = \frac{1}{e^{(E_b - \mu_b)/T} + 1} \qquad \text{and} \qquad f_{\bar{b}}(k) = \frac{1}{e^{(E_{\bar{b}} - \mu_{\bar{b}})/T} + 1}$$

[10] I say permit because CP violation is an evanescent effect. In the decay of particles, CP violation arises at one-loop if there is a quantum interference between a CP violating phase and a CP invariant phase. The decay amplitudes at one-loop of the particle and its antiparticles

$$i\mathscr{M} = \mathscr{A}_{tree} + e^{i\alpha_{CP}} \mathscr{A}_{one-loop}$$

and

$$i\bar{\mathscr{M}} = \mathscr{A}_{tree} + e^{-i\alpha_{CP}} \mathscr{A}_{one-loop}$$

give

$$|\mathscr{M}|^2 - |\bar{\mathscr{M}}|^2 \propto \sin\alpha_{CP} \times Im\left(\mathscr{A}_{tree} \mathscr{A}^*_{one-loop}\right)$$

which is, in practice, non-zero if the one-loop amplitude has a non-zero imaginary part. This happens when kinematics allows the particles within the loop to be on mass-shell.

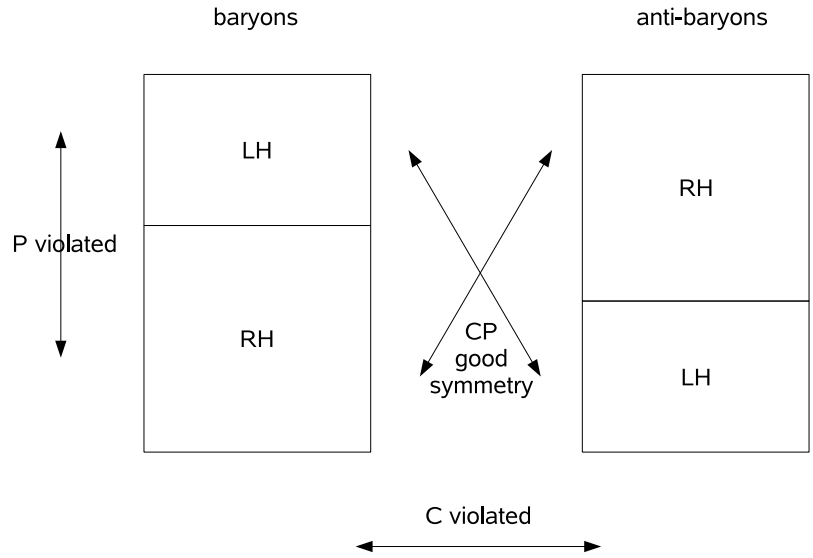


Fig. 29: Consider the decay of the putative particle Y. The area of each rectangle represents the number of baryons and antibaryons produced in the decay of an Y, $\bar{Y}$ pair. If P is broken, the number of left-handed (LH) and right-handed (RH) baryons is different. If C is violated the number of, say, RH baryons and LH antibaryons is different. However, if the combined symmetry CP is conserved the number of RH baryons and LH antibaryons is the same. The net result is no asymmetry. Note that P plays a secondary role here.

with $E_b = \sqrt{k^2 + m_b^2}$ and $E_{\bar{b}} = \sqrt{k^2 + m_{\bar{b}}^2}$, and μ_b and $\mu_{\bar{b}}$ are chemical potentials. CPT symmetry gives $m_b = m_{\bar{b}}$ while in equilibrium $b + \bar{b} \leftrightarrow \gamma + \gamma$ imposes

$$\mu_b = -\mu_{\bar{b}}.$$

If, moreover, processes which do not conserve B-number are in equilibrium, giving effectively $b + b \leftrightarrow \ldots + \gamma + \gamma$, then

$$\mu_b = \mu_{\bar{b}} = 0$$

Thus the final result is that

$$f_b(k) = f_{\bar{b}}(k)$$

and no asymmetry may be generated in thermodynamic equilibrium.

One possible scenario for departure from equilibrium is the following. Suppose that initially the Y and $\bar{Y}$ are in thermal equilibrium at some temperature $T > M_Y$, $n_Y = n_{\bar{Y}} \propto T^3$. The abundance is maintained by processes say like $Y + \bar{Y} \leftrightarrow x_{SM} + \bar{x}_{SM}$ with a rate Γ_A. Suppose that at $T \sim M_Y$, the annihilation rate drops below the expansion rate of the universe $\Gamma_A < H$. Then Y and $\bar{Y}$ are decoupled from the thermal bath and their abundance $n_Y/s = n_{\bar{Y}}/s$ stays constant instead of decreasing like $e^{-M_Y/T}$. If at some time after decoupling the Y and $\bar{Y}$ start to decay (this means that we have assumed that $\Gamma_D < \Gamma_A$) and no scattering processes like $b \leftrightarrow \bar{b}$ are in equilibrium ($\Gamma_S < H_{decay}$), a net baryon asymmetry

is produced which we may estimate to be of the order of

$$\frac{n_B}{s} \approx \frac{n_Y B_Y}{s} \sim \frac{B_Y}{g_*}\,,$$

where g_* is the number of degrees of freedom that are relativistic at the time of Y decay. The right way to do things nowadays is to write and solve a set of Boltzmann equations but the argument above gives the flavour of baryogenesis.

The scenario discussed above is nice but nowadays GUT baryogenesis is not much in favours. This is in part because SM anomalous processes that might erase a baryon asymmetry may have been in thermal equilibrium all the way down to the electroweak phase transition at $T \sim 1$ TeV. One way around is to use the fact that anomalous processes actually violate not just B but rather $B+L$.

Imagine there is initially a leptonic asymmetry $L_i \neq 0$ and that $B-L$ is conserved. Then SM anomalous processes in equilibrium in the early universe may partially convert this lepton asymmetry into a baryon asymmetry

$$B_f = -\frac{1}{2}L_i \qquad \text{and} \qquad L_f = \frac{1}{2}L_i\,.$$

This is a bit naive but a more refined derivation gives a similar conclusion (*i.e.*, a net baryon asymmetry is generated). This idea is at the basis of leptogenesis, a scenario by which first a lepton asymmetry is generated and then the lepton asymmetry is partially converted into a baryon asymmetry. The lepton asymmetry is typically believed to be generated through the CP violating decay of a heavy Majorana neutrino of mass M_R. Being a Majorana it may decay into a SM lepton l or SM antilepton $\bar{l}$. If CP is violated, the branching ratio $r_l \neq r_{\bar{l}}$ and a net lepton asymmetry is produced on average in the decay of the heavy Majorana. Decay after $T \sim M_R$ requires

$$\Gamma_D \sim \frac{\lambda^2}{4\pi} M_R \lesssim H(M_R) \sim g_*^{1/2}\frac{M_R^2}{M_{pl}} \quad \longrightarrow \quad M_R \gtrsim \frac{\lambda^2}{4\pi g_*^{1/2}} M_{pl}\,.$$

The bound on the Majorana mass depends on the value of the unknown coupling λ. Interestingly, heavy Majorana neutrinos are invoked to explain the smallness of SM neutrinos, through the see-saw mechanism (see the lectures by P. Hernandez at this school), with

$$m_\nu \sim \frac{\lambda^2 v^2}{M_R}$$

with $v = 246$ GeV, which gives

$$\frac{\Gamma_D}{H} \sim \frac{m_\nu M_{pl}}{4\pi v^2}\,.$$

Hence small SM neutrino masses go in the direction of having out-of-equilibrium decay of heavy Majorana neutrinos. There are many variations around this idea but, typically the required mass scale is

$$10^8\,\text{GeV} \lesssim M_R \lesssim 10^{15}\,\text{GeV}\,.$$

This is a very high scale, an unfortunate but quite generic feature of baryogenesis scenarios[11].

We summarize here what we have learned about the composition of the universe in the previous sections. We set $h^2 \approx 1/2$.

[11] There are many scenarios of baryogenesis. For the possibility of creating the baryon asymmetry around the electroweak scale, see [16].

Constituent	Fraction of Ω today	Origin
Photons	$1.25 \cdot 10^{-5}$	COBE measurement of CMBR temperature
Neutrinos	$10^{-3} \lesssim \Omega_{\nu,0} \lesssim 1.3 \cdot 10^{-1}$	Neutrino oscillations, lower bound Tritium decay, upper bound
Baryons	0.05	Primordial nucleosynthesis (CMB anistropies)
Dark matter	0.25	(Large-scale structure)
Dark energy	0.70	SnIa Hubble diagram
Curvature	0	(CMB anisotropies)

The topics in parenthesis will be covered in the last two sections.

7 Formation of large-scale structures

So far we have considered a universe that is perfecly homogeneous and isotropic. This is, however, a first approximation and we would like also to address the fact that there are inhomogeneities on various scales in the universe, like galaxies, clusters of galaxies and beyond. Inhomogeneities may be characterized by the density contrast $\Delta = \delta\rho/\rho$ where ρ is the average energy density of the fluid being considered. We saw in the first lecture (see Fig. 6) that $\Delta \ll 1$ on large scales, say larger than a few tens of Mpc. On smaller scales, $\Delta \gtrsim 1$ and non-linear effects are expected to be important. This is the scale of galaxies and clusters of galaxies, and their formation is a topic which is way beyond the scope of these lectures. On larger scales, however, a linear analysis should be applicable. The physics is thus fairly simple but as we shall see, it already tells us a great deal about the universe. The most important lessons of this section will be (1) that the mechanism that underlies the formation of large-scale structures is simply gravitational collapse and (2) that in an expanding universe, we need primordial inhomogeneities. In this way, we shall learn something about important cosmological parameters.

We shall continue to describe the content of the universe with simple fluids, like baryons, photons, or dark matter. The basics equations are those of perfect fluids. Let us consider for simplicity a non-relativistic fluid describes by its density $\rho(\vec{x},t)$ and velocity field $\vec{v}$. In the presence of gravity, these equations are

1. Continuity equation or conservation of mass gives

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0\,.$$

2. Euler equation which is the equivalent of Newton for a fluid is

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi\,.$$

where p is the pressure and Φ is the Newtonian potential.

3. Poisson equation gives the Newtonian gravitational potential

$$\nabla^2\Phi = 4\pi G\rho\,.$$

We linearize these equations, first assuming that the background is static (fluid at rest $\vec{v} = 0$), with density $\bar{\rho}$ and look for the equations for small perturbations.

$$\rho \to \bar{\rho} + \delta\rho \qquad \vec{v} \to \delta\vec{v} \qquad p \to \bar{p} + \delta p \qquad \Phi \to \bar{\Phi} + \delta\Phi$$

There is a bit of inconsistency here in assuming a static background, since $\bar{\rho} \neq 0 \rightarrow \vec{\nabla}\bar{\Phi} \neq 0$ and the latter is a source for $\vec{v}$. This is an old issue, already known to Newton, that a static uniform distribution of matter is unstable with respect to gravitational interactions, as we shall see.

Keeping the leading terms, the continuity equation and Euler give

$$\frac{\partial \delta\rho}{\partial t} + \bar{\rho}\vec{\nabla}\cdot\delta\vec{v} = 0$$

and

$$\frac{\partial \delta\vec{v}}{\partial t} = -\frac{1}{\bar{\rho}}\vec{\nabla}\delta p - \vec{\nabla}\delta\Phi \equiv -\frac{1}{\bar{\rho}}v_s^2\vec{\nabla}\delta\rho - \vec{\nabla}\delta\Phi,$$

where we have used $\partial p/\partial\rho = v_s^2$, where v_s is the speed of sound in the fluid. Taking the time derivative of the continuity equation and the divergence of Euler to eliminate $\delta\vec{v}$ allows us to write an equation for the density contrast $\Delta = \delta\rho/\bar{\rho}$

$$\frac{\partial^2\Delta}{\partial t^2} - v_s^2\nabla^2\Delta = \nabla^2\delta\Phi \equiv 4\pi G\bar{\rho}\Delta.$$

This is the familiar wave equation for a fluid disturbance, with phase velocity v_s, in presence of gravity. Plane wave solutions with wavenumber k have two independent solutions

$$\Delta \propto e^{\mp i\omega t + i\vec{k}\cdot\vec{x}}$$

where ω satisfies the dispersion relation

$$\omega = \sqrt{v_s^2k^2 - 4\pi G\bar{\rho}}.$$

The behaviour of these solutions depends on the sign of the expression in the square root. It is convenient to introduce the *Jeans wavenumber*

$$k_J = \sqrt{\frac{4\pi G\bar{\rho}}{v_s^2}}.$$

For $k > k_J$, corresponding to a regime in which gravity may be neglected compared to pressure, ω is real and inhomogeneities $\delta\rho$ behave as sound waves with $c = \omega/k = v_s\sqrt{1-k_J^2/k^2}$. If $k < k_J$, ω is imaginary and pressure can not prevent an inhomogeneity from growing exponentially, the signature of an instability,

$$\Delta \propto e^{\pm|\omega|t}.$$

The Jeans wavenumber k_J has a simple interpretation. It comes from the ratio of two time scales. On dimensional grounds, the time characteristic for gravity to act is given by $\tau_G \sim 1/\sqrt{G\rho}$ ('collapse time'), while pressure effects act a time scale $\tau_p \sim \lambda/v_s$. The condition $\tau_G \sim \tau_p$ gives $\lambda_J = 2\pi/k_J \sim v_s/\sqrt{G\rho}$.

The phenomenon of Jeans instability described above is at the core of the theory of large-scale structures in cosmology. It may give the impression that inhomogeneities may grow from infinitesimal perturbations. However, the expansion of the universe (the background) changes things in a crucial way. Taking into account expansion gives (see Kolb and Turner)

$$\frac{\partial^2\Delta}{\partial t^2} + 2H\frac{\partial\Delta}{\partial t} + \left(\frac{v_s^2k^2}{a^2} - 4\pi G\rho\right)\Delta = 0.$$

There is an extra term, linear in the Hubble parameter H. This is the analog of a friction term. Also we are using comoving coordinates, so that k (comoving wavenumber) is fixed and $k_{physical} = k/a$. For large wavenumbers (small scales), the solutions are oscillatory (with an amplitude that is decreasing because of the friction term). For small wavenumbers (large scales $v_sk/a \ll 1/H$) we may neglect the k^2 in the

equation (neglect pressure). Let us consider for simplicity a matter dominated, flat universe. Using the Friedmann equation, we may write

$$\frac{\partial^2\Delta}{\partial t^2} + 2H\frac{\partial\Delta}{\partial t} - 3/2H^2\Delta = 0\,.$$

We search for solutions of the form $\Delta \propto t^\alpha$. Inserting the differential equation and using $H = 2/3t$ gives

$$\alpha(\alpha-1) + 4/3\alpha - 2/3 = 0\,.$$

This equation has two solutions. One is $\alpha = -1$ or $\Delta_- \sim H$ is decreasing. The other one has $\alpha = 2/3$ or $\Delta_+ \propto t^{2/3}$ and is growing. This is important so we emphasize

$$\boxed{\Delta_+ \sim t^{2/3} \propto a(t)}$$

We see that the effect of expansion is to give a milder, power law behaviour to the instable solution. This is intuitively reasonable as the source of the instability is diluted by expansion, $\sim \rho \propto 1/t^2$ [12].

The moderate growth of Δ (in the linear regime) in an expanding universe implies that inhomogeneities had to be substantial (*ie* instead of infinitesimal if instabilities were growing exponentially) in the past to give rise to the large-scale structures seen today in the universe. One likely solution to this initial conditions problem is inflation. We shall see in the last section that a phase of accelerated expansion may give rise to a spectrum of primordial inhomogeneities which is consistent with observations.

A powerful way to probe these primordial inhomogeneities is to analyse anisotropies in the CMBR. This is because in the early universe hydrogen was ionized and the electrons were free. Through Thomson scattering

$$\gamma + e^- \leftrightarrow \gamma + e^-$$

which coupled electrons and photons, and through Coulomb scattering

$$e^- + p \leftrightarrow e^- + p$$

which coupled electrons and protons (we neglect the contribution of helium in our discussion here), the photons and baryons were effectively strongly coupled, a sort of photon-baryon fluid. As long as this coupling is effective, that is, as long as hydrogen is substantially ionized, we expect inhomogeneities in the density of baryons and in the energy density photons to be related. Since $\rho_b \propto T^3$ (baryons are non-relativistic) and $\rho_\gamma \propto T^4$,

$$\Delta_b = 3\Theta = \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$$

where

$$\Theta = \frac{\delta T}{T}\,.$$

Hence inhomogeneities in matter should be reflected in temperature inhomogeneities in the CMBR.

For the same reason that we can only see the edge of clouds, we may only observed the photons that were released around the time when the universe became transparent. This moment is called recombination or sometimes, and more appropriately, last scattering. Naively this took place when the average energy of photons $\sim T \propto a^{-1}$ was of the order of the binding energy of an electron in hydrogen $T \sim 13.6$

[12] There is a neat mechanical analogy. Consider a thin, long stick and put it vertically on your finger, trying to keep it straight. If you do not move your hand, it falls down rapidly (exponential instability). If you simultaneously let your hand fall down (diluting gravity), the stick falls more slowly (power law instability).

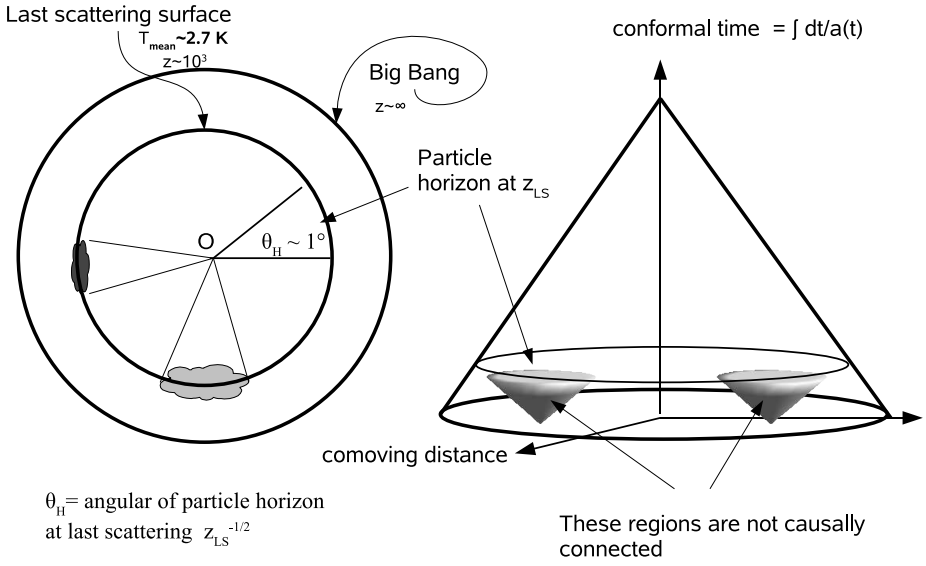


Fig. 30: Two views of the universe using conformal time η and comoving distances χ. We are at the centre of the circle today (LHS) or a the tip of the cone (RHS). We may see photons from as far back as the time of last scattering. The pictures also show the size of the particle horizon at the time of the last scattering, how far information may have propagated between the Big Bang and z_{LS}.

eV. However, very much like in our discussion of primordial nucleosynthesis in Section 4, there are many more photons than protons and electrons in the universe $n_b/n_\gamma \sim 10^{-9}$ and these many photons may easily ionize hydrogen[13]. This competititon between energy and entropy ensures that recombination takes place at a much lower temperature $T \sim 3000$ K ~ 0.25 eV, corresponding to a redshift $z_{LS} \sim 10^3$.

In the sequel, we will make use of conformal time

$$\eta = \int dt/a\,.$$

One motivation is that photons travel a distance η in a time interval η, since

$$ds^2 = dt^2 - a^2 d\chi^2 \equiv 0 \rightarrow ds^2 = a^2(d\eta^2 - d\chi^2)\,.$$

Hence the lapse of conformal time since the Big Bang is

$$\eta_0 = \int_0^1 \frac{dt}{a} \equiv d_{PH}$$

were d_{PH} is the distance to the horizon we first met in Section 3. Using η and χ coordinates, the causal structure of the universe takes then a very simple form, illustrated in Fig. 30.

Before last scattering, the photon-baryon fluid is essentially described by a single equation for Θ, which, neglecting the effect of gravity, is simply

$$\ddot{\Theta} + v_s^2 k^2 \Theta = 0$$

[13] A further complication is that recombination of an electron in the fundamental is accompanied by the emission of a photon which may in turn ionize another hydrogen atom, with no net effect. In practice, recombination goes (essentially) through the $2s$ state which may relax to the fundamental $1s$ through two photons.

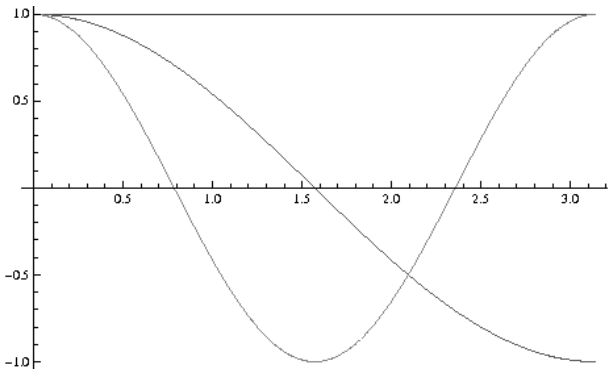


Fig. 31: Evolution in time $\Theta(\eta)$ until η_{ls} for $k \ll 1/c_s\eta_{ls}$, $k = \pi/c_s\eta_{ls}$ and $k = 2\pi/c_s\eta_{ls}$

where dots mean the derivative is taken with respect to the conformal time. This equation reflects the fact that pressure is important as long as photons and baryons are strongly coupled and so gravitational collapse of baryons is not possible. Instead the photon-baryon fluid undergoes acoustic oscillations, with corresponding heating and cooling of the fluid. This stops at last scattering, $\eta = \eta_{LS}$, when baryons and photons decouple. While pressure drops and baryons may start collapsing, the photons travel freely, carrying information about their temperature at last scattering which will be observed today as anisotropies in the CMBR temperature (see Fig. 30). Assuming $\dot{\Theta}(0) = 0$ (a prediction of inflation), the solution of the wave equation is

$$\Theta(v_s n_{LS}) = \Theta(0)\cos(kv_s\eta_{LS})\,.$$

Solutions for different k as a function of conformal time are shown in Fig. 31. Modes with $kv_s \ll \eta_{LS}$ do not evolve much before η_{LS}. Since η_{LS} is the comoving size of the horizon at last scattering, these correspond to perturbation on very large scales, larger than the size of the horizon at η_{LS}. Increasing k, the mode may undergo more and more oscillations. Those with their wavenumber satisfying

$$k_n = \frac{n\pi}{v_s\eta_{LS}}$$

with n integer, correspond to maximum/minimum of Δ at last scattering. This is seen in Fig. 32 which shows Θ^2 at last scattering as a function of k.

It is convenient to decompose temperature fluctuations in the CMBR in spherical harmonics, the analog on the sphere of Fourier modes,

$$\Theta_{LS}(\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Theta_{lm}Y_{lm}(\theta,\phi)\,.$$

A fluctuation on the scale $\lambda \sim 1/k$ sustains an angle approximately

$$\theta \approx \lambda/D\,,$$

where D is the comoving distance to the time of last scattering. To good approximation, $D \approx \eta_0$. Since l is conjugate to the angle θ, roughly $l \sim Dk$. Hence the peaks in Θ^2 should correspond to peaks at

$$l_n \approx n\frac{D\pi}{v_s\eta_{LS}}$$

in the power spectrum of the signal Θ,

$$\langle\Theta_l^2\rangle \equiv (2l+1)C_l$$

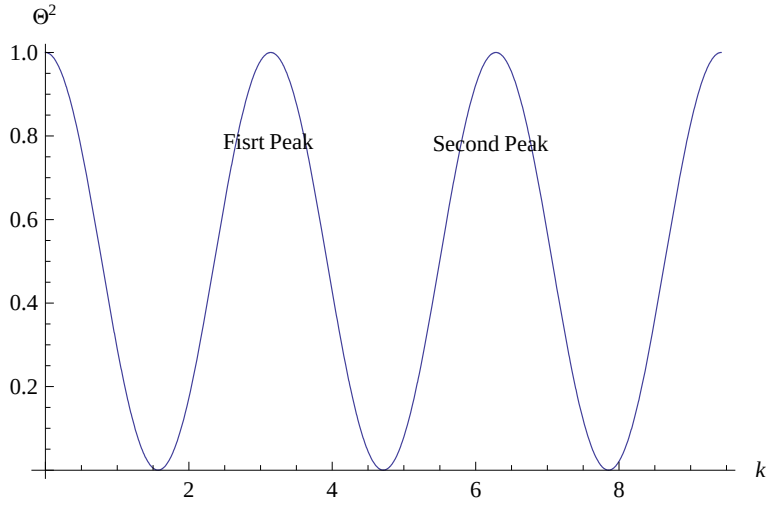


Fig. 32: Θ^2 (normalized to one) at η_{LS} as a function of k (normalized to $v_s\eta_{LS}$). There is a succession of peaks at $k_n = n\pi/v_s\eta_{LS}$.

obtained by averaging over m numbers[14].

Assuming that the universe is flat, matter dominated since last scattering $a \propto t^{2/3}$ gives $a \propto \eta^2$ and thus

$$\frac{\eta_0}{\eta_{LS}} = \frac{1}{\sqrt{1+z_{LS}}} \approx \frac{1}{30}$$

corresponding to an angle of about 2° and $l_1 \sim 200$ (using $v_s \sim 1/\sqrt{3}$ for the photon-baryon fluid). A compilation of data is shown in Fig. 33. With some imagination, one may recognize the peaks seen in Fig. 32. We may right away learn two things from these data.

1. The position of the first peak is related to the size of the horizon at last scattering. Assume you know the latter. By measuring the position of the peaks we actually measure the angular size that the horizon sustains on the sky. If the geometry is spherical, the angle would be larger than in a flat universe as Fig. 34 suggests. Correspondingly, the first peak would be shifted to the left (smaller l corresponds to larger θ on the sky). The opposite would occur for an hyperbolic geometry. Data are consistent with a flat universe.

2. On the largest scale, the CMBR anisotropies probe primordial inhomogeneities (not affected by local processes). We know actually since COBE that $\Theta \sim 10^{-5}$ (small l limit of the figure). Now, remember that we have seen that matter inhomogeneities grow like

$$\Delta_b \propto a = \frac{1}{1+z}$$

in a flat, matter-dominated universe. On large scales, we see structures today that have $\Delta_b = \mathscr{O}(1)$ on scales $\mathscr{O}(100\ \mathrm{Mpc})$, which at the time of last scattering correspond to scales that are beyond the horizon. We would thus expect $\Theta = \frac{1}{3}\Delta_b \approx 10^{-3}$ at $z = z_{LS}$, substantially larger than the amplitude inhomogeneities observed[15].

[14] If the anisotropies are statistically isotropic, the information in the m's is redundant, or rather say, gives an independent sample of the power in mode l. Of course, on large angular scales (small l) the sample is small and uncertainty comparitively large. This is the basis of the so-called 'cosmic variance' which leads to a large error on how accurately the power spectrum may be measured $\Delta C_l \propto C_l/\sqrt{2l+1}$.

[15] Strictly speaking we should also take into account the fact that if there were only baryons, the universe would be open. It is possible to show that the growth of inhomogeneities slows down when the expansion becomes curvature dominated, an effect

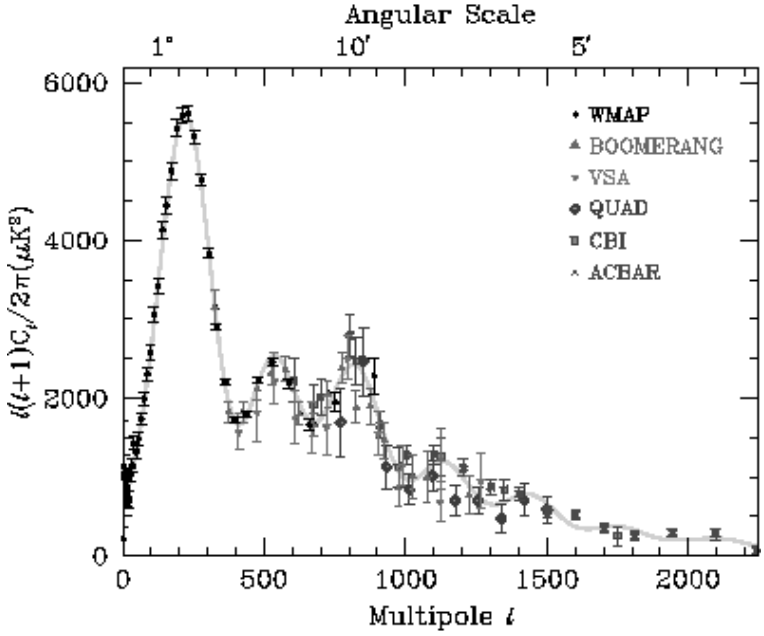
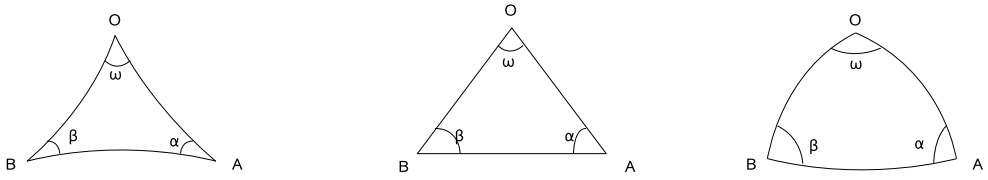


Fig. 33: Observation of the power spectrum of fluctuations in the CMBR, together with a theoretical prediction with an $\Omega_{dm} \approx 0.25$ and $\Omega_b \approx 0.05$ flat universe. From ref. [11].



$\theta_H < \theta_H|_{flat}$ $\theta_H = \theta_H|_{flat}$ $\theta_H > \theta_H|_{flat}$

Fig. 34: In a flat universe, the horizon at last scattering (supposed to be known) sustains an angle θ_H on the sky. If the geometry is spherical, light rays are bent inward and the angle would appear larger. The opposite is the case if the geometry is hyperbolic.

This turns out to be a strong indication for the existence of cold dark matter. The picture is (see Fig. 35) as follows. There is dark matter and it is composed of particles that do not interact with photons, baryons, and electrons. These particles were non-relativistic at the time when there was equality between the energy density in radiation and that in matter a_{eq} (this is necessary because in a radiation-dominated inhomogeneities only grow logarithmically). At a_{EQ} this dark matter, which is non-interacting and thus feels no pressure, collapses as $\Delta_{dm} \propto a(t)$. In the meantime, photons and baryons are strongly coupled and, until a_{LS}, undergo acoustic oscillations. At last scattering, the baryons become free and they may fall in the gravitational potential of the dark matter. End of story.

The power spectrum of Fig. 33 is quite different from the naive form of Fig. 32. Explaining all this in detail would take us way beyond these lectures (see Refs. [17], [18] or [19]). There is one feature

which makes the conclusion that fluctuations are too small in a universe with only baryonic matter even more dramatic. See Kolb and Turner.

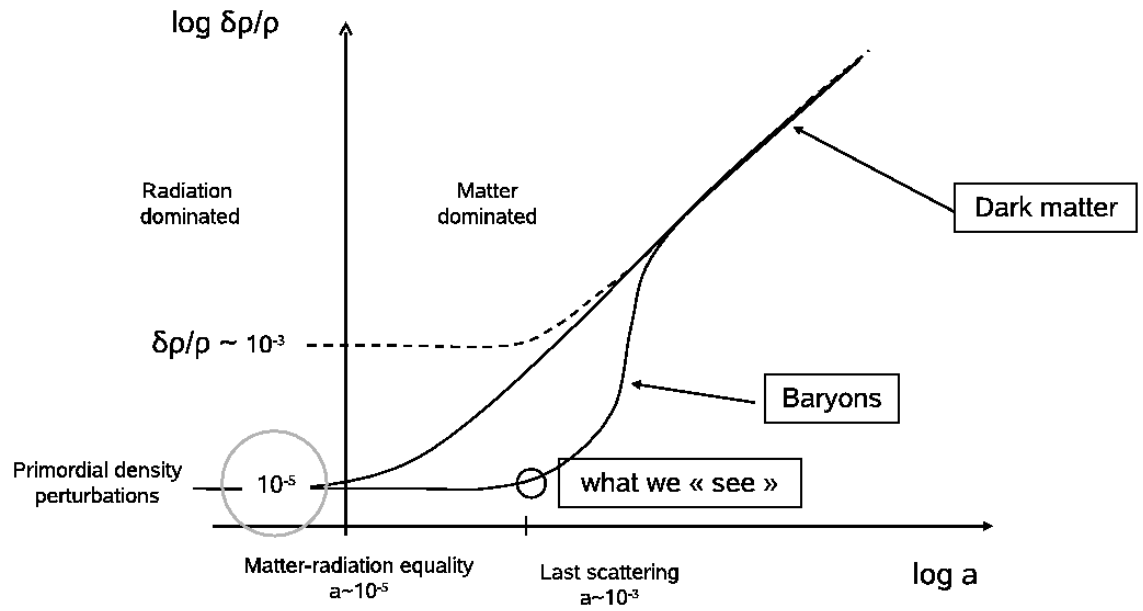


Fig. 35: Schematic evolution of dark matter, baryons, and photons between matter-radiation equality and today.

that I would like to emphasize, however, which is the effect of baryons on the relative height of the peaks.

If we take into account gravity and baryons, the equation for Θ becomes slightly more complicated:

$$(1+3\rho_b/4\rho_\gamma)\ddot{\Theta}+k^2v_s^2\Theta \approx -(1+3\rho_b/4\rho_\gamma)k^2v_s^2\Phi\,.$$

where Φ is the Newtonian gravitational potential. Let us discus first the impact of introducing Φ, so assume $\rho_b \ll \rho_\gamma$. It so happens that Φ may be taken to be constant in first approximation. Then the equation may be rewritten as

$$\ddot{\Theta}_{eff}+k^2v_s^2\Theta_{eff}=0\,,$$

with $\Theta_{eff}=\Theta+\Phi$. The meaning of Θ_{eff} is the following. Imagine that a photon of frequency ν is in a gravitational potential well $\Phi<0$ at last scattering. In climbing from the gravitational well, it loses energy and its frequency is redshifted by a factor $\delta\nu/\nu=\Phi$. Since $\delta\nu/\nu\equiv\delta T/T$, the Newtonian potential will manifests itself as a temperature fluctuation on top of the intrinsic temperature fluctuation Θ. Hence the observable quantity is the combination Θ_{eff}. The relevant solution to the equation for Θ_{eff} is

$$\Theta_{eff}(\eta)=\Theta_{eff}(0)\cos(kv_s\eta)\,.$$

This is called the Sachs-Wolfe effect[16]. Note that to a potential well hence a region of higher density, $\Phi<0$, corresponds a lower temperature.

[16] If Φ is constant after recombination, potential wells on the way of photons between last scattering and us have no net effect as the losses are compensated exactly by gains and the other way around. If, however, Φ has some time dependence, as is the case if the universe becomes dominated by a cosmological constant, then there is an extra contribution to the temperatures

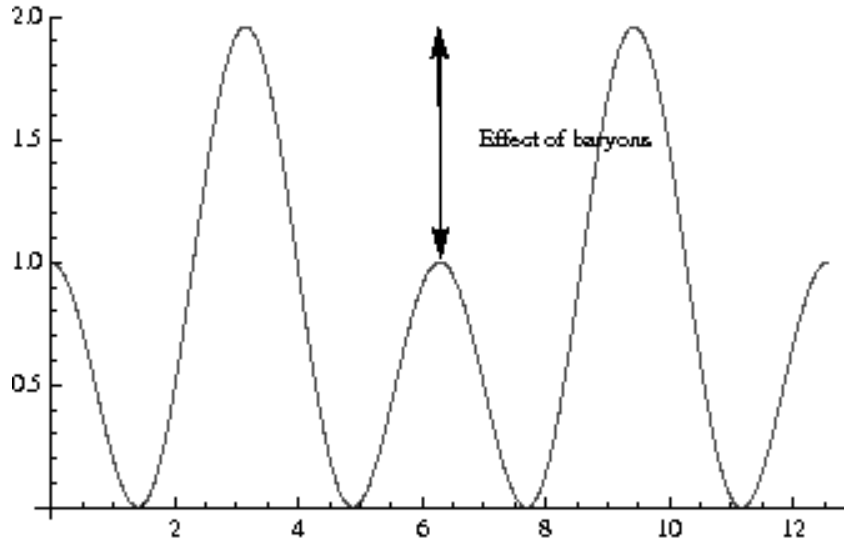


Fig. 36: Impact of baryons on Θ^2_{eff}.

Now consider the effect of baryons, $\rho_b/n_\gamma \neq 0$. The effect of baryons is to reduce the sound velocity $v_s \to c_s = v_s/\sqrt{1+3\rho_b/4\rho_\gamma}$ and to shift the origin of oscillations of Θ_{eff}. As the observed temperature fluctuation is still Θ_{eff} the solution becomes

$$\Theta_{eff}(\eta) = (\Theta_{eff}(0) + 3\rho_b/4\rho_\gamma\Phi)\cos(kc_s\eta) - 3\rho_b/4\rho_\gamma\Phi\,.$$

For $\Phi < 0$ (attractive well), the net effect of the shift is to lower the even peaks and to raise the odd ones, with a difference between peaks $\propto \rho_b/\rho_\gamma$, Fig. 36. Intuitively, baryons tend to accumulate in a potential well and to increase compression peaks (odd peaks). Hence the difference between the first at $l \sim 200$ and the second peak at $l \sim 400$ in CMBR anisotropies data gives a measurement of ρ_b/ρ_γ and this of the baryon asymmetry of the universe.

Many other cosmological parameters may be extracted from analysis of CMBR anisotropies, together with input from large-scale surveys. Of particular interest to high-energy physicists are the constraints that may be put on neutrino masses. The constraints from WMAP are limited because neutrinos are a subdominant component of matter at the time of last scattering (see however the latest WMAP data release). Since neutrinos are very light, they have substantial momentum at the time of matter-radiation equality. Their motion prevents them from collapsing until the time they become non-relativistic. In the meantime they may propagate a distance λ_{FS} called the free streaming scale. If neutrinos were to constitute a substantial fraction of dark matter, no structure could form on scales $\lambda \lesssim \lambda_{FS}$. This is the imprint of so-called Hot Dark Matter. Observations indicate that dark matter is rather made of Cold Dark Matter (*i.e.*, a form of dark matter with little momentum at matter-radiation equality that may form structures on all scales) and puts a limit (that is large-scale surveys which probe smaller cosmological scales than the CMBR anisotropies) on neutrino masses (typically a fraction of eV depending which data are taken into account, see [20] and lectures by P.Hernandez at this school).

To conclude this section, I give a fair (albeit personal) summary of joint WMAP and other data below.

$$\begin{aligned} \Omega_b &= 0.04 \\ \Omega_{dm} &= 0.26 \end{aligned}$$

anisotropies called the integrated Sachs-Wolfe effect.

$$
\begin{aligned}
H &= 70\,\mathrm{km}\cdot s^{-1}\cdot \mathrm{Mpc}^{-1} \\
\Omega_0 &= 0.950 \\
\Omega &= 1 \\
\Omega_\Lambda &= 0.70 \\
\frac{p_\Lambda}{\rho_\Lambda} &= -1 \\
n &= 0.95
\end{aligned}
$$

The flat, cold dark matter plus cosmological constant or ΛCDM or *Concordance Model* is now the standard model of cosmology (see Fig. 22).

The parameter n in the list above is new to us. This is the spectral index, to be defined in the last section. It tells us that the spectrum of temperature fluctuations in the CMBR is nearly scale invariant. That the spectrum should be such is one of the predictions of inflation. That our universe should be spatially flat is anothter one. The most puzzling result is the presence of a cosmological constant, or at least the presence of a fluid that behaves like a cosmological constant (dark energy). A cosmological constant leads to accelerated expansion, one of the key features of inflation.

8 Inflation

An early phase of accelerated expansion or inflation does the following:

- It solves the flatness problem
- It solves the horizon problem
- It generates primordial inhomogeneities
- It predicts that the spectrum of inhomogeneities is scale invariant

All these features are consistent with observations. What caused inflation is so far unknown but it is easy to implement it in a phenomenological way and thus not challenges the Big Bang model [21].

We have not paid much attention to this feature, but it is quite puzzling that our universe is so flat. After all this is just one possible solution among all the possible geometries. There is also a more critical problem called the *flatness problem*. The problem is the following. Take the Friedmann equation and write it as

$$|1-\Omega| = \frac{|K|}{a^2H^2} \propto \dot{a}^{-2}\,.$$

From the Raychaudhuri equation (and our intuition of gravity)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,,$$

we know that for both a matter or radiation dominated universe $\dot{a}$ decreases. Hence we expect $|1-\Omega|$ to increase with time. For instance take $|1-\Omega| = \mathscr{O}(10^{-2})$ today. Then at $T_{EQ} \sim 30000$ K

$$|1-\Omega| = \mathscr{O}(10^{-6})$$

while at $T \sim 1$ MeV

$$|1-\Omega| = \mathscr{O}(10^{-18})\,.$$

You may go back further in the past. The conclusion is that the geometry of the universe had to be very very close to flat for the universe to appear flat today.

A simple remedy is to make the size of the universe very large, much larger than our horizon. We can achieve this dynamically if the universe goes through a phase of accelerated expansion, or inflation, since then

$$\ddot{a} > 0$$

and

$$|1-\Omega| \propto \dot{a}^{-2} \to 0$$

How long should inflation last? Assume that inflation is driven by a fluid such that $p \approx -\rho$ for some time. Then $H \approx$ constant and

$$a = a_i e^{H(t-t_i)}$$

Assume inflation happened at an energy scale $\mathscr{O}(10^{16})$ GeV. Taking

$$H \sim \frac{(10^{16}\,\mathrm{GeV})^2}{M_{\mathrm{Pl}}} \sim 10^{14}\,\mathrm{GeV} \qquad \text{and with} \qquad \Delta t \sim 10^7 t_{\mathrm{Pl}} \sim 10^{-36}\mathrm{s}$$

where t_{Pl} is the Planck time, we then see that the scale factor would have grown by a huge factor within a very short time

$$a_f/a_i \sim e^{100} \sim 10^{44}\,.$$

Compare this with $a_0/a_{LS} \sim 10^3$, the change of the scale factor between last scattering and today and that took about $13 \cdot 10^9$ years.

Another issue is why our universe is very uniform. Consider for instance the time of last scattering. We see essentially the same CMBR temperature (to within 10^5) in all directions. We have seen that the horizon at z_{LS} sustains an angle of about 2° on the sky. Physical conditions may be pretty uniform within the scale of the horizon, but how come that they are the same on larger scales? The largest distance we may probe is the distance to the horizon today,

$$d_H|_{today} \sim H_0^{-1} \approx 10^{28}\,\mathrm{cm}\,.$$

At $T \sim 10^{15}$ GeV the universe was much smaller. Our horizon today occupied a region of about $d \sim 10^{-28} H_0^{-1} \sim 1$ cm. All our universe within one cubic cm. This was small, but is actually much larger than the particle horizon at that time, which was

$$d_H \sim H^{-1} \sim 10^{-14}\,\mathrm{GeV} \sim 10^{-28}\,\mathrm{cm}\,.$$

Inflating this small distance by a factor of $10^{28} \sim e^{65}$ would give a simple solution to the horizon or homogeneity problem. This is illustrated in Fig. 37.

The simplest model of inflation posits the existence of a scalar field ϕ with potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2\,.$$

Suppose that the scalar field is initially shifted away from its minimum and that it has a small kinetic energy. Also assume it is homogeneous. The stress-energy tensor of the scalar field

$$T^\mu_{\ \nu} = \partial^\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathscr{L}$$

where $\mathscr{L}$ is the Lagrangian density, takes the form of that of a perfect fluid

$$T^\mu_{\ \nu} = \mathrm{diag}(\rho, -p, -p, -p)$$

with energy density ρ and pressure p:

$$\rho \approx \frac{1}{2}\dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \approx \frac{1}{2} m^2 \phi^2$$

and

$$p \approx \frac{1}{2}\dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 \approx -\frac{1}{2} m^2 \phi^2$$

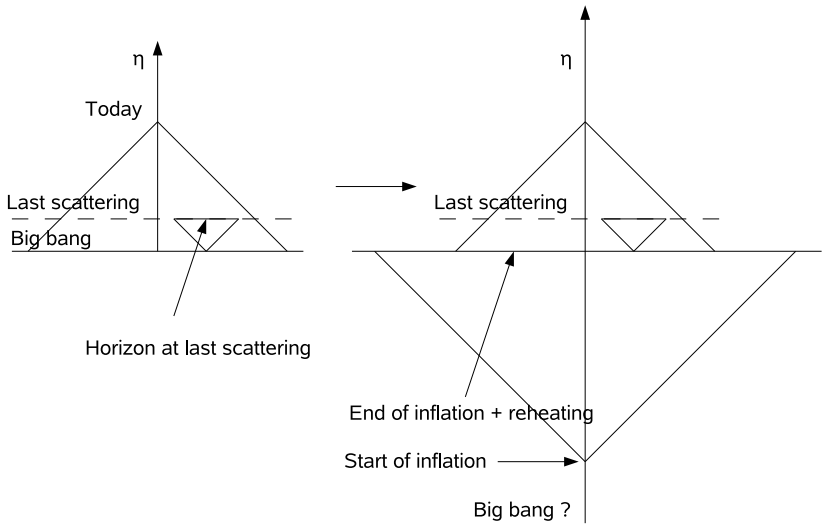


Fig. 37: How inflation solves (RHS) the horizon problem (LHS). Note that the lapse of physical time (t) since inflation is about $13 \cdot 10^9$ years while that of inflation itself lasts an instant.

or

$$p \approx -\rho\,.$$

This is all we need for inflation to occur. There are many variations around this simple scheme (called chaotic inflation —the name comes from the initial conditions necessary to initiate inflation—) but the basic idea stays the same. There is some field, called the inflaton, that evolves slowly so that its potential energy is larger than its kinetic energy. Contrary to what one may think, that the field is slowly evolving does not require much tuning. This is because the equation of motion of our scalar field in an expanding universe is

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} + m^2\phi = 0\,.$$

The effect of expansion is in the friction term. If the energy density is dominated by the potential of the scalar field

$$H^2 \sim Gm^2\phi^2 \equiv \frac{m^2\phi^2}{m_{pl}^2}\,.$$

If $\phi \gg m_{\rm Pl}$ initially, the friction term is dominant over the second derivative and the field rolls down slowly. This stops at roughly $\phi \sim m_{pl}$, at which points the kinetic energy of the scalar field is no longer negligible and inflation stops. Eventually the scalar field oscillates around its minimum. The universe after inflation is very big but also very cold as everything, including any thermal bath (or baryon number for that matter), has been diluted by the exponential growth of the size of the universe. It is expected that the inflaton is coupled to SM fields (or its siblings) and that its energy stored in oscillations may be transformed in heat. How this *reheating* takes place is a complex problem and is not fully understood yet.

This is so far a classical process. However, the accelerated expansion of the universe during inflation also has a quantum manifestation. This effect is a bit analogous to the phenomenon of pair

production in the presence of a strong electric field. It is also closely related to the Hawking radiation of black holes. The details are beyond the scope of these lectures but let me give you the flavour. Consider some massless scalar field χ during inflation. Its equation of motion, using comoving Fourier modes, is

$$\frac{\partial^2\chi}{\partial t^2} + 2H\frac{\partial\chi}{t} + \frac{k^2}{a^2}\chi = 0\,.$$

It is convenient to use conformal time $dt = ad\eta$. Then the equation becomes

$$\ddot{\chi} + 2aH\dot{\chi} + k^2\chi = 0\,.$$

During inflation $H \approx$ const and let $a \approx e^{Ht}$[17]. The conformal time may be then expressed as

$$\eta = -\frac{1}{H}e^{-Ht} \qquad \text{and} \qquad aH \equiv -\frac{1}{\eta}\,.$$

Now we eliminate the friction term by using a field redefinition $\chi = \eta v(\eta)$. The equation becomes finally

$$\ddot{v} + \left(k^2 - \frac{2}{\eta^2}\right)v = 0\,.$$

This is like the equation of an harmonic oscillator. Initially we may have $k^2\eta^2 \gg 1$ and the solutions are simply oscillations, like in vacuum. However, as time goes by, $k^2\eta^2$ decreases (remark $|\eta| \to 0$ toward the future) and the equation becomes that of a reversed oscillator, the landmark of an instability. The quantization of this system leads to the conclusion that, during inflation, modes are created out of the vacuum (like electron-positron pairs may be created by a strong electromagnetic field). While $\langle\chi\rangle = 0$, the correlator of χ, which is equivalent to the power spectrum, is non-vanishing:

$$P_\chi(k) = \langle\chi^2\rangle \propto \frac{H^2}{k^3}\,.$$

For $H \sim$ constant, the spectrum of fluctuations in χ is scale-invariant *i.e.*

$$\langle\chi^2(x)\rangle \propto \int d^3k P_\phi(k) \propto \int \frac{dk}{k}H^2$$

in the sense that there is the same power per log interval of k.

A similar result holds for fluctuations of the scalar field that triggers inflation or inflaton ϕ. The discussion is, however, made complicated by the fact that fluctuation in a scalar quantity is, in general, not invariant under general coordinate transformations. However, the essence of the story is that fluctuations generated in the inflaton field may be expressed as fluctuations in the Newtonian potential for modes larger than the size of the horizon. The fluctuation in the inflaton may disappear but fluctuations in the Newtonian potential survive (they stay constant), and these in turn leave their imprint on the dark matter. The spectrum of fluctuations is predicted to be (nearly) scale invariant. This feature, called the Harrison-Zeldovich spectrum, is supported by both the CMB and the large-scale structure surveys.

9 Epilogue

Accelerated expansion is easy to implement, but difficult to comprehend. For instance, just addding a constant V_0 to the potential of a scalar field gives a contribution to its stress-energy tensor

$$\delta T^\mu_\nu = V_0\delta^\mu_\nu\,,$$

[17] Different normalization than in the rest of the lectures.

which is equivalent to adding a cosmological constant

$$\rho = -p = V_0 \,.$$

To agree with observation we could decide that V_0 is zero (or very small) but the problem is that the cosmological constant strikes back at the quantum level. Indeed, in quantum field theories we are effectively dealing with harmonic oscillators, labelled by momentum, with zero energy $1/2\,\omega \equiv 1/2\sqrt{k^2+m^2}$ (or $-1/2\,\omega$ for fermions) per degree of freedom. Then a naive summation over the modes of a scalar field

$$\delta V_{\text{quantum}} \propto \sum_k \frac{1}{2}\sqrt{k^2+m^2}$$

gives a divergent result, $\delta V_{q.} \to \infty$. In quantum field theory we usually discard these contributions because only energy differences matter when we compute cross-sections or discuss symmetry breaking. However $\delta V_{q.}$ has weight and it is not clear on which basis we may get rid of it if we take into account gravity. If instead we assume that the summation over modes is cut off at the Planck energy scale, we get

$$\delta V_{q.} \equiv \delta\rho_\Lambda \sim m_{\text{Pl}}^4$$

This is about 120 orders of magnitude than what is observed, the current accelerated expansion of the universe giving

$$\Omega_\Lambda \approx 0.70 \quad \to \quad \rho_\Lambda \approx (2\cdot 10^{-3}\,\text{eV})^4 \,.$$

This so-called cosmological constant problem is one of the biggest issue in fundamental physics (see for instance Ref. [22] for a review)[18].

Another puzzling facet of the cosmological constant problem is that the Ω_Λ observed is close but not equal to one. Since the contribution to Ω of a true cosmological constant was negligible until recently but will be dominant in the near future, $\Omega_\Lambda \lesssim 1$ means that today is a special moment in the history of the universe, see Fig. 38. This so-called coincidence problem has motivated the construction of dynamical models of cosmological constant or dark energy models. Their equation of state generically departs from that of a true cosmological constant, a feature that may be constrained by further studies of the Hubble diagram and large-scale structure.

It is, however, fair to say that the relation between most models of dark energy and more fundamental principles (like, say, string theory or quantum gravity) is rather loose and the situation regarding the nature of dark energy is likely to stay unsettled for some time.

Acknowledgements

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[18] Perhaps things would be simpler if the cosmological constant was observed to be zero, for that could mean that some sort of symmetry principle makes harmless the contributions of quantum fields to vacuum energy. Supersymmetry is such a symmetry principle, for the contribution of bosons $E_B \sum_B 1/2\omega_B$ and fermions $E_F = -\sum_F 1/2\omega_F$ compensate each other if supersymmetry is not broken. However, supersymmetry, if it exists, has to be broken to be consistent with observations and the cosmological constant problem remains.

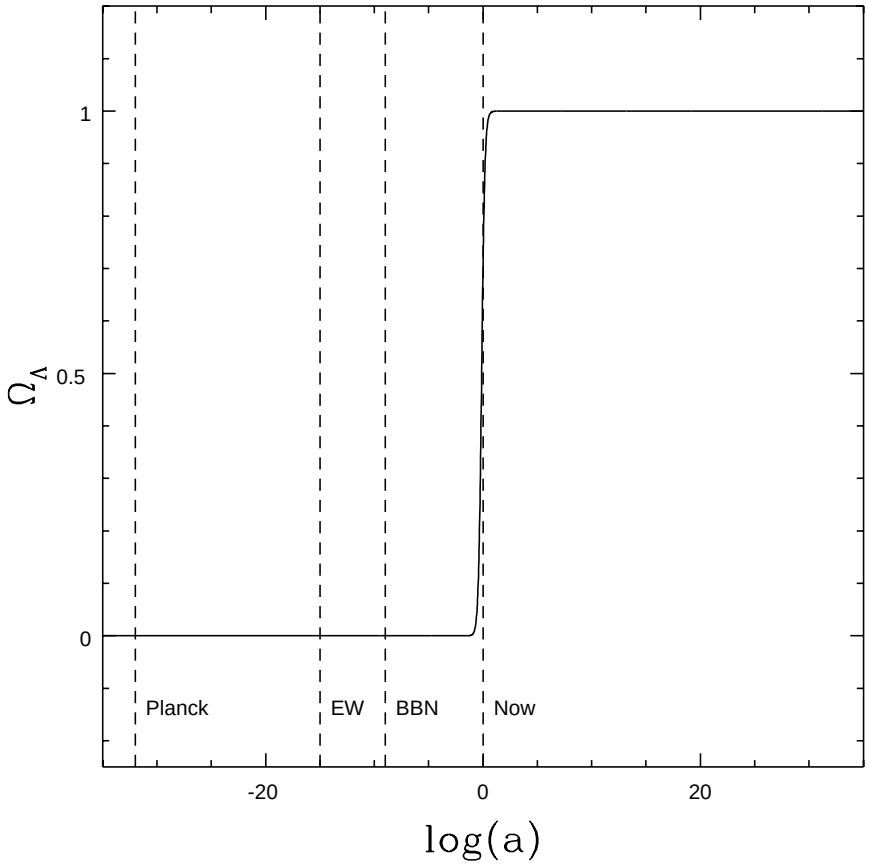

Fig. 38: Evolution Ω_Λ as a function of the scale factor. It emphasizes the impression (in $\log(a)$) that we live at a very special moment in the history of the universe [22].

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Appendices

I. Conversion factors

$$\begin{aligned} 1\,\mathrm{GeV} &\equiv 1.6022\cdot 10^{-10}\,\mathrm{J} \\ &\equiv 1.1605\cdot 10^{13}\,\mathrm{K} \\ &\equiv 1.7827\cdot 10^{-27}\mathrm{kg} \end{aligned}$$

$$\begin{aligned} 1\,\mathrm{GeV}^{-1} &= 1.9733\cdot 10^{-16}\,\mathrm{m} \\ &= 6.6522\cdot 10^{-25}\,\mathrm{s} \end{aligned}$$

$$\begin{aligned} 1cm &\equiv 5.068\cdot 10^{13}\,\mathrm{GeV}^{-1} \\ 1s &\equiv 1.519\cdot 10^{24}\,\mathrm{GeV}^{-1} \\ 1g &\equiv 5.608\cdot 10^{23}\mathrm{GeV} \end{aligned}$$

$$\begin{aligned} 1\mathrm{AU} &= 1.496\cdot 10^{11}\,\mathrm{m}\ \text{(Astronomical Unit)} \\ 1\mathrm{pc} &= 3.086\cdot 10^{16}\,\mathrm{m}\ \text{(parsec)} \\ 1\mathrm{year} &= 3.156\cdot 10^{7}\,\mathrm{s} \\ 1'' &= 4.85\cdot 10^{-6}\,\mathrm{rad} \end{aligned}$$

II. Some cosmological parameters

$$m_{\mathrm{Pl}} = \frac{1}{\sqrt{G_N}} = 2.2\times 10^{-5} g = 1.2209\cdot 10^{19} \qquad \text{reduced Planck mass}$$

$$t_{\mathrm{Pl}} = 5.4\times 10^{-44}\,\mathrm{s} \qquad \text{Planck time}$$

$$
\begin{aligned}
l_{\rm Pl} &= 1.6\times10^{-33}\,\mathrm{cm} \qquad \text{Planck length}\\[2ex]
H_0 &= 100\,h\,\mathrm{km\cdot s^{-1}\cdot Mpc^{-1}}\\[2ex]
H_0^{-1} &= 9.78h^{-1}\,\mathrm{Gyr} = 2998h^{-1}\,\mathrm{Mpc}\\
\rho_{c0} &= 3M_{\rm Pl}^2H_0^2 = 2.775h^{-1}\times10^{11}\frac{\mathrm{M_\odot}}{(h^{-1}\,\mathrm{Mpc})^3} \qquad \text{critical density}\\
&= 1.88h^2\times10^{-29}g\cdot cm^{-3} = (3\times10^{-3}eV)^4h^2\\
&= 10.5h^2\,\mathrm{GeV\cdot m^{-3}}\\[2ex]
\Omega_{\gamma,0}h^2 &= 2.47\times10^{-5} \qquad \text{photon density parameter}\\
\rho_{\gamma0} &= 4.61\cdot10^{-34}(T_{\gamma0}/2.725\,\mathrm{K})^4\ g\cdot\mathrm{cm^{-3}}\\
n_{\gamma0} &= 410\,(T_{\gamma0}/2.725\,\mathrm{K})^3\ \mathrm{cm^{-3}}\\[2ex]
n_{\nu0} &= 3/11n_{\gamma0} = 113\,(T_{\gamma0}/2.725\,\mathrm{K})^3\ \mathrm{cm^{-3}}\\
h^2\Omega_{\nu0} &= \frac{\sum_i m_{\nu_i}}{94\,\mathrm{eV}}(T_{\gamma0}/2.725\,\mathrm{K})^3\\[2ex]
\Omega_{R,0}h^2 &= 4.17\times10^{-5} \qquad \text{three massless neutrinos}\\[2ex]
\eta &= n_{b0}/n_{\gamma0}\approx6\cdot10^{-10}
\end{aligned}
$$